INTRODUCTION TO THE 2012 KAPLAN SCHWESER STUDY NOTES FRM EXAM PART II

Thank you for trusting Kaplan Schweser to help you reach your career and academic goals. We are very pleased to be able to help you prepare for the 2012 FRM Exam. In this introduction, I want to explain what is included in the Study Notes, suggest how you can best use Kaplan Schweser materials to prepare for the exam, and direct you toward other educational resources you will find helpful as you study for the exam.

Study Notes—A 4-book set that includes complete coverage of all risk-related topic areas and AIM statements, as well as Concept Checkers (multiple-choice questions for every assigned reading) and Challenge Problems (exam-like questions). At the end of each book, we have included relevant questions from past GARP FRM practice exams. These old exam questions are a great tool for understanding the format and difficulty of actual exam questions.

To help you master the FRM material and be well prepared for the exam, we offer several additional educational resources, including:

8-Week Online Class—Live online program (eight 3-hour sessions) that is offered each week, beginning in March for the May exam and September for the November exam. The online class brings the personal attention of a classroom into your home or office with 24 hours of real-time instruction led by either Dr. John Paul Broussard, CFA, FRM, PRM or Dr. Greg Filbeck, CFA, FRM, CAIA. The class offers in-depth coverage of difficult concepts, instant feedback during lecture and Q&A sessions, and discussion of past FRM exam questions. Archived classes are available for viewing at any time throughout the study season. Candidates enrolled in the Online Class also have access to downloadable slide files and Instructor E-mail Access, where they can send questions to the instructor at any time.

If you have purchased the Schweser Study Notes as part of the Essential, Premium, or PremiumPlus Solution, you will also receive access to Instructor-led Office Hours. Office Hours allow you to get your FRM-related questions answered in real time and view questions from other candidates (and faculty answers) as well. Office Hours is a text-based, live, interactive, online chat with the weekly online class instructor. Archives of previous Instructor-led Office Hours sessions are sorted by topic and are posted shortly after each session.

Practice Exams—The Practice Exam Book contains two full-length, 80-question (4-hour) exams. These exams are important tools for gaining the speed and confidence you will need to pass the exam. Each exam contains answer explanations for self-grading. Also, by entering your answers at Schweser.com, you can use our Performance Tracker to find out how you have performed compared to other Kaplan Schweser FRM candidates.
Interactive Study Calendar—Use your Online Access to tell us when you will start and what days of the week you can study. The Interactive Study Calendar will create a study plan just for you, breaking each topic area into daily and weekly tasks to keep you on track and help you monitor your progress through the FRM curriculum.

Online Question Database—In order to retain what you learn, it is important that you quiz yourself often. We offer download and online versions of our FRM SchweserPro Qbank, which contains hundreds of practice questions and explanations for Part II of the FRM Program.

In addition to these study products, there are many educational resources available at Schweser.com, including the FRM Video Library and the FRM Exam-tips Blog. Just log into your account using the individual username and password that you received when you purchased the Schweser Study Notes.

How to Succeed

The FRM exam is a formidable challenge, and you must devote considerable time and effort to be properly prepared. You must learn the material, know the terminology and techniques, understand the concepts, and be able to answer at least 70% of the questions quickly and correctly. 250 hours is a good estimate of the study time required on average, but some candidates will need more or less time depending on their individual backgrounds and experience. To provide you with an overview of the FRM Part II curriculum, we have included a list of all GARP assigned readings in the order they appear in our Study Notes. Every topic in our Notes is cross-referenced to an FRM assigned reading, so should you require additional clarification with certain concepts, you can consult the appropriate assigned reading.

There are no shortcuts to studying for this exam. Expect GARP to test you in a way that will reveal how well you know the FRM curriculum. You should begin studying early and stick to your study plan. You should first read the Study Notes and complete the Concept Checkers for each topic. At the end of each book, you should answer the provided Challenge Problems and practice exam questions to understand how concepts have been tested in the past. You can also attend our 8-Week Online Class to assist with retention of the exam concepts. You should finish the overall curriculum at least two weeks before the FRM exam. This will allow sufficient time for Practice Exams and further review of those topics that you have not yet mastered.

Best wishes for your studies and your continued success,

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Study Guide

Market Risk Measurement and Management
Part II Exam Weight: 25%

1: Chapter 3 – Estimating Market Risk Measures
2: Chapter 4 – Non-parametric Approaches
3: Chapter 5 – Modeling Dependence: Correlations and Copulas – Appendix
4: Chapter 7 – Parametric Approaches (II): Extreme Value

5: Chapter 6 – Backtesting VaR
6: Chapter 11 – VaR Mapping

7: Chapter 6 – Measures of Price Sensitivity Based on Parallel Yield Shifts
8: Chapter 7 – Key Rate and Bucket Exposures
9: Chapter 9 – The Science of Term Structure Models

10: Chapter 19 – Volatility Smiles
11: Chapter 25 – Exotic Options

12: Chapter 1 – Overview of Mortgages and the Consumer Mortgage Market

13: Chapter 8 – Basics of Residential Mortgage Backed Securities

Frank Fabozzi, Anand Bhattacharya, William Berliner, Mortgage Backed Securities, 2nd Edition
14: Chapter 2 – Overview of the Mortgage-Backed Securities Market
15: Chapter 10 – Techniques for Valuing MBS

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CREDIT RISK MEASUREMENT AND MANAGEMENT

Part II Exam Weight: 25%


16: Chapter 23 – Credit Risk


17: Chapter 4 – Extending the VaR Approach to Non-tradable Loans


18: Chapter 3 – Default Risk: Quantitative Methodologies
19: Chapter 4 – Loss Given Default


20: Chapter 6 – Portfolio Effects: Risk Contributions and Unexpected Losses


21: “Measuring and Marking Counterparty Risk” by Eduardo Canabarro and Darrell Duffie


22: Chapter 6 – Pricing and Hedging Counterparty Risk: Lessons Re-Learned?, by Eduardo Canabarro


23: Chapter 18 – Credit Risks and Credit Derivatives


24: Chapter 12 – Credit Derivatives and Credit-Linked Notes
25: Chapter 13 – The Structuring Process
26: Chapter 17 – Cash Collateralized Debt Obligations
John Hull, Options, Futures, and Other Derivatives, 8th Edition.
27: Chapter 24 – Credit Derivatives

Christopher Culp, Structured Finance and Insurance: The Art of Managing Capital and Risk
28: Chapter 16 – Securitization

29: Adam Ashcroft and Til Schuermann, “Understanding the Securitization of Subprime Mortgage Credit,” Federal Reserve Bank of New York Staff Reports, no. 318 (March 2008).

OPERATIONAL AND INTEGRATED RISK MANAGEMENT
Part II Exam Weight: 25%

30: Chapter 14 – Capital Allocation and Performance Measurement


32: Chapter 16 – Model Risk

33: Chapter 16 – Liquidity, the Ultimate Operational Risk
34: Chapter 17 – Analyzing and Measuring Liquidity Risk
35: Chapter 18 – Funding Risk
36: Chapter 19 – Managing and Mitigating Liquidity Risks

37: Chapter 14 – Estimating Liquidity Risks


RISK MANAGEMENT AND INVESTMENT MANAGEMENT
Part II Exam Weight: 15%


48: Chapter 14 – Portfolio Construction


49: Chapter 24 - Portfolio Performance Evaluation


51: Chapter 17 – Risk Monitoring and Performance Measurement


52: Chapter 7 – Portfolio Risk: Analytical Methods
53: Chapter 17 – VaR and Risk Budgeting in Investment Management


54: Chapter 6 – Risk Budgeting for Pension Funds and Investment Managers Using VaR, by Michelle McCarthy


55: Chapter 11 – Overview of Hedge Funds
56: Chapter 12 – Hedge Fund Investment Strategies
57: Chapter 16 – Overview of Private Equity


**CURRENT ISSUES IN FINANCIAL MARKETS**

*Part II Exam Weight: 10%*

62: Gregory Connor, Thomas Flavin, and Brian O’Kelly, “The U.S. and Irish Credit Crises: Their Distinctive Differences and Common Features.”


66: Chapter 4 – The Collapse of the Icelandic Banking System, by René Kallestrup and David Lando

67: Chapter 9 – Measuring and Managing Risk in Innovative Financial Instruments, by Stuart M. Turnbull

68: Chapter 20 – Active Risk Management: A Credit Investor’s Perspective, by Vineer Bhansali
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**ESTIMATING MARKET RISK MEASURES**

**Exam Focus**

In this topic, the focus is on the estimation of market risk measures, such as value at risk (VaR). VaR identifies the probability that losses will be greater than a pre-specified threshold level. For the exam, be prepared to evaluate and calculate VaR using historical simulation and parametric models (both normal and lognormal return distributions). One drawback to VaR is that it does not estimate losses in the tail of the returns distribution. Expected shortfall (ES) does, however, estimate the loss in the tail (i.e., after the VaR threshold has been breached) by averaging loss levels at different confidence levels. Coherent risk measures incorporate personal risk aversion across the entire distribution and are more general than expected shortfall. Quantile-quantile (QQ) plots are used to visually inspect if an empirical distribution matches a theoretical distribution.

**Estimating Returns**

To better understand the material in this topic, it is helpful to recall the computations of arithmetic and geometric returns. Note that the convention when computing these returns (as well as VaR) is to quote return losses as positive values. For example, if a portfolio is expected to decrease in value by $1 million, we use the terminology "expected loss is $1 million" rather than "expected profit is -$1 million."

**Profit/loss data:** Change in value of asset/portfolio, $P_t$, at the end of period $t$ plus any interim payments, $D_t$.

$$P/L_t = P_t + D_t - P_{t-1}$$

**Arithmetic return data:** Assumption is that interim payments do not earn a return (i.e., no reinvestment). Hence, this approach is not appropriate for long investment horizons.

$$r_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t + D_t}{P_{t-1}} - 1$$

**Geometric return data:** Assumption is that interim payments are continuously reinvested. Note that this approach ensures that asset price can never be negative.

$$R_t = \ln \left( \frac{P_t + D_t}{P_{t-1}} \right)$$

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Historical Simulation Approach

AIM 1.1: Estimate VaR using a historical simulation approach.

Estimating VaR with a historical simulation approach is by far the simplest and most straightforward VaR method. To make this calculation, you simply order return observations from largest to smallest. The observation that follows the threshold loss level denotes the VaR limit. We are essentially searching for the observation that separates the tail from the body of the distribution. More generally, the observation that determines VaR for $n$ observations at the $(1 - \alpha)$ confidence level would be: $(\alpha \times n) + 1$.

Professor's Note: Recall that the confidence level, $(1 - \alpha)$, is typically a large value (e.g., 95%) whereas the significance level, usually denoted as $\alpha$, is much smaller (e.g., 5%).

To illustrate this VaR method, assume you have gathered 1,000 monthly returns for a security and produced the distribution shown in Figure 1. You decide that you want to compute the monthly VaR for this security at a confidence level of 95%. At a 95% confidence level, the lower tail displays the lowest 5% of the underlying distribution's returns. For this distribution, the value associated with a 95% confidence level is a return of $-15.5\%$. If you have $1,000,000 invested in this security, the one-month VaR is $155,000$ ($-15.5\% \times 1,000,000$).

Figure 1: Histogram of Monthly Returns

![Histogram of Monthly Returns](image-url)
Example: Identifying the VaR limit

Identify the ordered observation in a sample of 1,000 data points that corresponds to VaR at a 95% confidence level.

Answer:

Since VaR is to be estimated at 95% confidence, this means that 5% (i.e., 50) of the ordered observations would fall in the tail of the distribution. Therefore, the 51st ordered loss observation would separate the 5% of largest losses from the remaining 95% of returns.

Professor’s Note: VaR is the quantile that separates the tail from the body of the distribution. With 1,000 observations at a 95% confidence level, there is a certain level of arbitrariness in how the ordered observations relate to VaR. In other words, should VaR be the 50th observation (i.e., \(\alpha \times n\)), the 51st observation (i.e., \((\alpha \times n) + 1\)), or some combination of these observations? In this example, using the 51st observation was the approximation for VaR, and the method used in the assigned reading. However, on past FRM exams, VaR using the historical simulation method has been calculated as just: \((\alpha \times n)\), in this case, as the 50th observation.

Example: Computing VaR

A long history of profit/loss data closely approximates a standard normal distribution (mean equals zero; standard deviation equals one). Estimate the 5% VaR using the historical simulation approach.

Answer:

The VaR limit will be at the observation that separates the tail loss with area equal to 5% from the remainder of the distribution. Since the distribution is closely approximated by the standard normal distribution, the VaR is 1.65 (5% critical value from the \(z\)-table). Recall that since VaR is a one-tailed test, the entire significance level of 5% is in the left tail of the returns distribution.

From a practical perspective, the historical simulation approach is sensible only if you expect future performance to follow the same return generating process as in the past. Furthermore, this approach is unable to adjust for changing economic conditions or abrupt shifts in parameter values.
Parametric Estimation Approaches

AIM 1.2: Estimate VaR using a parametric estimation approach assuming that the return distribution is either normal or lognormal.

In contrast to the historical simulation method, the parametric approach (e.g., the delta-normal approach) explicitly assumes a distribution for the underlying observations. For this AIM, we will analyze two cases: (1) VaR for returns that follow a normal distribution, and (2) VaR for returns that follow a lognormal distribution.

Normal VaR

Intuitively, the VaR for a given confidence level denotes the point that separates the tail losses from the remaining distribution. The VaR cutoff will be in the left tail of the returns distribution. Hence, the calculated value at risk is negative, but is typically reported as a positive value since the negative amount is implied (i.e., it is the value that is at risk). In equation form, the VaR at significance level $\alpha$ is:

$$\text{VAR}(\alpha \%) = -\mu_{P/L} + \sigma_{P/L} \times z_\alpha$$

where $\mu$ and $\sigma$ denote the mean and standard deviation of the profit/loss distribution and $z$ denotes the critical value (i.e., quantile) of the standard normal. In practice, the population parameters $\mu$ and $\sigma$ are not likely known, in which case the researcher will use the sample mean and standard deviation.

Example: Computing VaR (normal distribution)

Assume that the profit/loss distribution for XYZ is normally distributed with an annual mean of $15 million and a standard deviation of $10 million. Calculate the VaR at the 95% and 99% confidence levels using a parametric approach.

Answer:

$\text{VaR}(5\%) = -$15 million + $10 million \times 1.65 = $1.5 million. Therefore, XYZ expects to lose at most $1.5 million over the next year with 95% confidence. Equivalently, XYZ expects to lose more than $1.5 million with a 5% probability.

$\text{VaR}(1\%) = -$15 million + $10 million \times 2.33 = $8.3 million. Note that the VaR (at 99% confidence) is greater than the VaR (at 95% confidence) as follows from the definition of value at risk.

Now suppose that the data you are using is arithmetic return data rather than profit/loss data. The arithmetic returns follow a normal distribution as well. As you would expect, because of the relationship between prices, profits/losses, and returns, the corresponding VaR is very similar in format:

$$\text{VAR}(\alpha \%) = (-\mu_r + \sigma_r \times z_\alpha) \times P_{t-1}$$
Example: Computing VaR (arithmetic returns)

A portfolio has a beginning period value of $100. The arithmetic returns follow a normal distribution with a mean of 10% and a standard deviation of 20%. Calculate VaR at both the 95% and 99% confidence levels.

Answer:

\[
\text{VaR(5\%)} = (-10\% + 1.65 \times 20\%) \times 100 = 23.0 \\
\text{VaR(1\%)} = (-10\% + 2.33 \times 20\%) \times 100 = 36.6
\]

Lognormal VaR

The lognormal distribution is right-skewed with positive outliers and bounded below by zero. As a result, the lognormal distribution is commonly used to counter the possibility of negative asset prices (\(P_t\)). Technically, if we assume that geometric returns follow a normal distribution (\(\mu_R, \sigma_R\)), then the natural logarithm of asset prices follows a normal distribution and \(P_t\) follows a lognormal distribution. After some algebraic manipulation, we can derive the following expression for lognormal VaR:

\[
\text{VaR(\alpha\%)} = P_t \times \left( 1 - e^{\mu_R - \sigma_R \times \alpha} \right)
\]

Example: Computing VaR (lognormal distribution)

A diversified portfolio exhibits a normally distributed geometric return with mean and standard deviation of 10% and 20%, respectively. Calculate the 5% and 1% lognormal VaR assuming the beginning period portfolio value is $100.

Answer:

\[
\text{Lognormal VaR(5\%)} = 100 \times (1 - \exp[0.1 - 0.2 \times 1.65]) \\
= 100 \times (1 - \exp[-0.23]) \\
= 20.55
\]

\[
\text{Lognormal VaR(1\%)} = 100 \times (1 - \exp[0.1 - 0.2 \times 2.33]) \\
= 100 \times (1 - \exp[-0.366]) \\
= 30.65
\]
Note that the calculation of lognormal VaR (geometric returns) and normal VaR (arithmetic returns) will be similar when we are dealing with short-time periods and practical return estimates.

**Expected Shortfall**

AIM 1.3: Estimate expected shortfall given P/L or return data.

A major limitation of the VaR measure is that it does not tell the investor the amount or magnitude of the actual loss. VaR only provides the maximum value we can lose for a given confidence level. The expected shortfall (ES) provides an estimate of the tail loss by averaging the VaRs for increasing confidence levels in the tail. Specifically, the tail mass is divided into \( n \) equal slices and the corresponding \( n - 1 \) VaRs are computed. For example, if \( n = 5 \), we can construct the following table based on the normal distribution:

<table>
<thead>
<tr>
<th>Confidence level</th>
<th>VaR</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>96%</td>
<td>1.7507</td>
<td></td>
</tr>
<tr>
<td>97%</td>
<td>1.8808</td>
<td>0.1301</td>
</tr>
<tr>
<td>98%</td>
<td>2.0537</td>
<td>0.1729</td>
</tr>
<tr>
<td>99%</td>
<td>2.3263</td>
<td>0.2726</td>
</tr>
<tr>
<td>Average</td>
<td>2.003</td>
<td></td>
</tr>
<tr>
<td>Theoretical true value</td>
<td>2.063</td>
<td></td>
</tr>
</tbody>
</table>

Observe that the VaR increases (from Difference column) in order to maintain the same interval mass (of 1%) because the tails become thinner and thinner. The average of the four computed VaRs is 2.003 and represents the probability-weighted expected tail loss (a.k.a. expected shortfall). Note that as \( n \) increases, the expected shortfall will increase and approach the theoretical true loss [2.063 in this case; the average of a high number of VaRs (e.g., greater than 10,000)]

**Estimating Coherent Risk Measures**

AIM 1.4: Define coherent risk measures.

AIM 1.5: Describe the method of estimating coherent risk measures by estimating quantiles.

A more general risk measure than either VaR or ES is known as a coherent risk measure. A coherent risk measure is a weighted average of the quantiles of the loss distribution where the weights are user-specific based on individual risk aversion. ES (as well as VaR) is a special case of a coherent risk measure. When modeling the ES case, the weighting function is set to \( 1 / (1 - \text{confidence level}) \) for all tail losses. All other quantiles will have a weight of zero.
Under expected shortfall estimation, the tail region is divided into equal probability slices and then multiplied by the corresponding quantiles. Under the more general coherent risk measure, the entire distribution is divided into equal probability slices weighted by the more general risk aversion (weighting) function.

This procedure is illustrated for $n = 10$. First, the entire return distribution is divided into nine (i.e., $n - 1$) equal probability mass slices at 10%, 20%, ..., 90% (i.e., loss quantiles). Each breakpoint corresponds to a different quantile. For example, the 10% quantile (confidence level = 10%) relates to $-1.2816$, the 20% quantile (confidence level = 20%) relates to $-0.8416$, and the 90% quantile (confidence level = 90%) relates to $1.2816$. Next, each quantile is weighted by the specific risk aversion function and then averaged to arrive at the value of the coherent risk measure.

This coherent risk measure is more sensitive to the choice of $n$ than expected shortfall, but will converge to the risk measure's true value for a sufficiently large number of observations. The intuition is that as $n$ increases, the quantiles will be further into the tails where more extreme values of the distribution are located.

**AIM 1.6:** Describe the method of estimating standard errors for estimators of coherent risk measures.

Sound risk management practice reminds us that estimators are only as useful as their precision. That is, estimators that are less precise (i.e., have large standard errors and wide confidence intervals) will have limited practical value. Therefore, it is best practice to also compute the standard error for all coherent risk measures.

*Professor's Note: The process of estimating standard errors for estimators of coherent risk measures is quite complex, so your focus should be on interpretation of this concept.*

First, let's start with a sample size of $n$ and arbitrary bin width of $b$ around quantile, $q$. Bin width is just the width of the intervals, sometimes called “bins,” in a histogram. Computing standard error is done by realizing that the square root of the variance of the quantile is equal to the standard error of the quantile. After finding the standard error, a confidence interval for a risk measure such as VaR can be constructed as follows:

$$[q + se(q) \times z_{\alpha}] > VaR > [q - se(q) \times z_{\alpha}]$$
Example: Estimating standard errors

Construct a 90% confidence interval for 5% VaR (the 95th quantile) drawn from a standard normal distribution. Assume bin width = 0.1 and that the sample size is equal to 500.

Answer:

The quantile value, q, corresponds to the 5% VaR which occurs at 1.65 for the standard normal distribution. The confidence interval takes the following form:

\[ [1.65 + 1.65 \times \text{se}(q)] > \text{VaR} > [1.65 - 1.65 \times \text{se}(q)] \]

Professor's Note: Recall that a confidence interval is a two-tailed test (unlike VaR), so a 90% confidence level will have 5% in each tail. Given that this is equivalent to the 5% significance level of VaR, the critical values of 1.65 will be the same in both cases.

Since bin width is 0.1, q is in the range 1.65 ± 0.1/2 = [1.7, 1.6]. Note that the left tail probability, \( p \), is the area to the left of 1.7 for a standard normal distribution.

Next, calculate the probability mass between [1.7, 1.6], represented as \( f(q) \). From the standard normal table, the probability of a loss greater than 1.7 is 0.045 (left tail). Similarly, the probability of a loss less than 1.6 (right tail) is 0.945. Collectively, \( f(q) = 1 - 0.045 - 0.945 = 0.01 \)

The standard error of the quantile is derived from the variance approximation of \( q \) and is equal to:

\[ \text{se}(q) = \sqrt{\frac{p(1-p)}{f(q)}} \]

Now we are ready to substitute in the variance approximation to calculate the confidence interval for VaR:

\[ 1.65 + 1.65 \sqrt{\frac{0.045(1 - 0.045)}{500}} > \text{VaR} > 1.65 - 1.65 \sqrt{\frac{0.045(1 - 0.045)}{500}} \]

\[ = 3.18 > \text{VaR} > 0.12 \]

Let's return to the variance approximation and perform some basic comparative statistics. What happens if we increase the sample size holding all other factors constant? Intuitively, the larger the sample size the smaller the standard error and the narrower the confidence interval.
Now suppose we increase the bin size, $h$, holding all else constant. This will increase the probability mass $f(q)$ and reduce $p$, the probability in the left tail. The standard error will decrease and the confidence interval will again narrow.

Lastly, suppose that $p$ increases indicating that tail probabilities are more likely. Intuitively, the estimator becomes less precise and standard errors increase, which widens the confidence interval. Note that the expression $p(1 - p)$ will be maximized at $p = 0.5$.

The above analysis was based on one quantile of the loss distribution. Just as the previous section generalized the expected shortfall to the coherent risk measure, we can do the same for the standard error computation. Thankfully, this complex process is not the focus of the AIM statement.

**Quantile-Quantile Plots**

AIM 1.7: Describe the use of QQ plots for identifying the distribution of data.

A natural question to ask in the course of our analysis is, “From what distribution is the data drawn?” The truth is that you will never really know since you only observe the realizations from random draws of an unknown distribution. However, visual inspection can be a very simple but powerful technique.

In particular, the quantile-quantile (QQ) plot is a straightforward way to visually examine if empirical data fits the reference or hypothesized theoretical distribution (assume standard normal distribution for this discussion). The process graphs the quantiles at regular confidence intervals for the empirical distribution against the theoretical distribution. As an example, if both the empirical and theoretical data are drawn from the same distribution, then the median (confidence level = 50%) of the empirical distribution would plot very close to zero, while the median of the theoretical distribution would plot exactly at zero.

Continuing in this fashion for other quantiles (40%, 60%, and so on) will map out a function. If the two distributions are very similar, the resulting QQ plot will be linear.

Let us compare a theoretical standard normal distribution relative to an empirical $\epsilon$-distribution (assume that the degrees of freedom for the $\epsilon$-distribution are sufficiently small and that there are noticeable differences from the normal distribution). We know that both distributions are symmetric, but the $\epsilon$-distribution will have fatter tails. Hence, the quantiles near zero (confidence level = 50%) will match up quite closely. As we move further into the tails, the quantiles between the $\epsilon$-distribution and the normal will diverge (see Figure 3). For example, at a confidence level of 95%, the critical $z$-value is $-1.65$, but for the $\epsilon$-distribution, it is closer to $-1.68$ (degrees of freedom of approximately 40). At 99% confidence, the difference is even larger, as the $z$-value is equal to $-1.96$ and the $\epsilon$-stat is equal to $-2.02$. More generally, if the middles of the QQ plot match up, but the tails do not, then the empirical distribution can be interpreted as symmetric with tails that differ from a normal distribution (either fatter or thinner).
Figure 3: QQ Plot
1. Historical simulation is the easiest method to estimate value at risk. All that is required is to reorder the profit/loss observations in increasing magnitude of losses and identify the breakpoint between the tail region and the remainder of distribution.

2. Parametric estimation of VaR requires a specific distribution of prices or equivalently, returns. This method can be used to calculate VaR with either a normal distribution or a lognormal distribution.

3. Under the assumption of a normal distribution, VaR (i.e., delta-normal VaR) is calculated as follows:
   \[ \text{VaR} = -\mu_{P/L} + \sigma_{P/L} \times z_{\alpha} \]

   Under the assumption of a lognormal distribution, lognormal VaR is calculated as follows:
   \[ \text{VaR} = P_{t-1} \times \left(1 - e^{\mu_R - \sigma_R x_{\alpha}}\right) \]

4. VaR identifies the lower bound of the profit/loss distribution, but it does not estimate the expected tail loss. Expected shortfall overcomes this deficiency by dividing the tail region into equal probability mass slices and averaging their corresponding VaRs.

5. A coherent risk measure is a weighted average of the quantiles of the loss distribution where the weights are user-specific based on individual risk aversion. A coherent risk measure will assign each quantile (not just tail quantiles) a weight. The average of the weighted VaRs is the estimated loss.

6. Sound risk management requires the computation of the standard error of a coherent risk measure to estimate the precision of the risk measure itself. The simplest method creates a confidence interval around the quantile in question. To compute standard error, it is necessary to find the variance of the quantile, which will require estimates from the underlying distribution.

7. The quantile-quantile (QQ) plot is a visual inspection of an empirical quantile relative to a hypothesized theoretical distribution. If the empirical distribution closely matches the theoretical distribution, the QQ plot would be linear.
1. The VaR at a 95% confidence level is estimated to be 1.56 from a historical simulation of 1,000 observations. Which of the following statements is most likely true?
   A. The parametric assumption of normal returns is correct.
   B. The parametric assumption of lognormal returns is correct.
   C. The historical distribution has fatter tails than a normal distribution.
   D. The historical distribution has thinner tails than a normal distribution.

2. Assume the profit/loss distribution for XYZ is normally distributed with an annual mean of $20 million and a standard deviation of $10 million. The 5% VaR is calculated and interpreted as which of the following statements?
   A. 5% probability of losses of at least $3.50 million.
   B. 5% probability of earnings of at least $3.50 million.
   C. 95% probability of losses of at least $3.50 million.
   D. 95% probability of earnings of at least $3.50 million.

3. Which of the following statements about expected shortfall estimates and coherent risk measures are true?
   A. Expected shortfall and coherent risk measures estimate quantiles for the entire loss distribution.
   B. Expected shortfall and coherent risk measures estimate quantiles for the tail region.
   C. Expected shortfall estimates quantiles for the tail region and coherent risk measures estimate quantiles for the non-tail region only.
   D. Expected shortfall estimates quantiles for the entire distribution and coherent risk measures estimate quantiles for the tail region only.

4. Which of the following statements most likely increases standard errors from coherent risk measures?
   A. Increasing sample size and increasing the left tail probability.
   B. Increasing sample size and decreasing the left tail probability.
   C. Decreasing sample size and increasing the left tail probability.
   D. Decreasing sample size and decreasing the left tail probability.

5. The quantile-quantile plot is best used for what purpose?
   A. Testing an empirical distribution from a theoretical distribution.
   B. Testing a theoretical distribution from an empirical distribution.
   C. Identifying an empirical distribution from a theoretical distribution.
   D. Identifying a theoretical distribution from an empirical distribution.
Cross Reference to GARP Assigned Reading – Dowd, Chapter 3

Concept Checker Answers

1. D The historical simulation indicates that the 5% tail loss begins at 1.56, which is less than the 1.65 predicted by a standard normal distribution. Therefore, the historical simulation has thinner tails than a standard normal distribution.

2. D The value at risk calculation at 95% confidence is: $-20 \text{ million} + 1.65 \times 10 \text{ million} = -$3.50 million. Therefore, XYZ is expected to lose at least -3.50 million over the next year. Since the expected loss is negative, the interpretation is that XYZ will earn less than 3.50 million with 5% probability, which is equivalent to XYZ earning at least $3.50 million with 95% probability.

3. B ES estimates quantiles for n – 1 equal probability masses in the tail region only. The coherent risk measure estimates quantiles for the entire distribution including the tail region.

4. C Decreasing sample size clearly increases the standard error of the coherent risk measure given that standard error is defined as:

\[
\text{se}(q) = \sqrt{\frac{p(1-p)}{n} f(q)}
\]

As the left tail probability, p, increases, the probability of tail events increases, which also increases the standard error. Mathematically, \( p(1-p) \) increases as \( p \) increases until \( p = 0.5 \). Small values of \( p \) imply smaller standard errors.

5. C Once a sample is obtained, it can be compared to a reference distribution for possible identification. The QQ plot maps the quantiles one to one. If the relationship is close to linear, then a match for the empirical distribution is found. The QQ plot is used for visual inspection only without any formal statistical test.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP. This topic is also covered in:

## Non-Parametric Approaches

### Exam Focus

This topic introduces non-parametric estimation and bootstrapping (i.e., resampling). The key difference between these approaches and parametric approaches discussed in the previous topic is that with non-parametric approaches the underlying distribution is not specified, and it is a data driven, not assumption driven, analysis. For example, historical simulation is limited by the discreteness of the data, but non-parametric analysis "smoothes" the data points to allow for any VaR confidence level between observations. For the exam, pay close attention to the description of the bootstrap historical simulation approach as well as the various weighted historical simulations approaches.

Non-parametric estimation does not make restrictive assumptions about the underlying distribution like parametric methods, which assume very specific forms such as normal or lognormal distributions. Non-parametric estimation lets the data drive the estimation. The flexibility of these methods makes them excellent candidates for VaR estimation, especially if tail events are sparse.

### Bootstrap Historical Simulation Approach

AIM 2.1: Describe the bootstrap historical simulation approach to estimating coherent risk measures.

The bootstrap historical simulation is a simple and intuitive estimation procedure. In essence, the bootstrap technique draws a sample from the original data set, records the VaR from that particular sample and "returns" the data. This procedure is repeated over and over and records multiple sample VaRs. Since the data is always "returned" to the data set, this procedure is akin to sampling with replacement. The best VaR estimate from the full data set is the average of all sample VaRs.

This same procedure can be performed to estimate the expected shortfall (ES). Each drawn sample will calculate its own ES by slicing the tail region into \( n \) slices and averaging the VaRs at each of the \( n - 1 \) quantiles. This is exactly the same procedure described in the previous topic. Similarly, the best estimate of the expected shortfall for the original data set is the average of all of the sample expected shortfalls.

Empirical analysis demonstrates that the bootstrapping technique consistently provides more precise estimates of coherent risk measures than historical simulation on raw data alone.
**USING NON-PARAMETRIC ESTIMATION**

**AIM 2.2:** Describe historical simulation using non-parametric density estimation.

The clear advantage of the traditional historical simulation approach is its simplicity. One obvious drawback, however, is that the discreteness of the data does not allow for estimation of VaRs between data points. If there were 100 historical observations, then it is straightforward to estimate VaR at the 95% or the 96% confidence levels, and so on. However, this method is unable to incorporate a confidence level of 95.5%, for example. More generally, with $n$ observations, the historical simulation method only allows for $n$ different confidence levels.

One of the advantages of non-parametric density estimation is that the underlying distribution is free from restrictive assumptions. Therefore, the existing data points can be used to "smooth" the data points to allow for VaR calculation at all confidence levels. The simplest adjustment is to connect the midpoints between successive histogram bars in the original data set's distribution. See Figure 1 for an illustration of this *surrogate density function*. Notice that by connecting the midpoints, the lower bar "receives" area from the upper bar, which "loses" an equal amount of area. In total, no area is lost, only displaced, so we still have a probability distribution function, just with a modified shape. The shaded area in Figure 1 represents a possible confidence interval, which can be utilized regardless of the size of the data set. The major improvement of this non-parametric approach over the traditional historical simulation approach is that VaR can now be calculated for a continuum of points in the data set.

**Figure 1: Surrogate Density Function**

Following this logic, one can see that the linear adjustment is a simple solution to the interval problem. A more complicated adjustment would involve connecting curves, rather than lines, between successive bars to better capture the characteristics of the data.
WEIGHTED HISTORICAL SIMULATION APPROACHES

AIM 2.3: Describe the following weighted historic simulation approaches:

- Age-weighted historic simulation
- Volatility-weighted historic simulation
- Correlation-weighted historic simulation
- Filtered historical simulation

The previous weighted historical simulation, discussed in Topic 1, assumed that both current and past (arbitrary) \( n \) observations up to a specified cutoff point are used when computing the current period VaR. Older observations beyond the cutoff date are assumed to have a zero weight and the relevant \( n \) observations have equal weight of \((1 / n)\). While simple in construction, there are obvious problems with this method. Namely, why is the \( n \)th observation as important as all other observations, but the \((n + 1)\)th observation is so unimportant that it carries no weight? Current VaR may have “ghost effects” of previous events that remain in the computation until they disappear (after \( n \) periods). Furthermore, this method assumes that each observation is independent and identically distributed. This is a very strong assumption, which is likely violated by data with clear seasonality (i.e., seasonal volatility). This topic identifies four improvements to the traditional historical simulation method.

**Age-weighted Historic Simulation**

The obvious adjustment to the equal-weighted assumption used in historical simulation is to weight recent observations more and distant observations less. One method proposed by Boudoukh, Richardson, and Whitelaw is as follows. Assume \( w(1) \) is the probability weight for the observation that is one day old. Then \( w(2) \) can be defined as \( \lambda w(1) \), \( w(3) \) can be defined as \( \lambda^2 w(1) \), and so on. The decay parameter, \( \lambda \), can take on values \( 0 \leq \lambda \leq 1 \) where values close to 1 indicate slow decay. Since all of the weights must sum to 1, we conclude that \( w(1) = (1 - \lambda) / (1 - \lambda^n) \). More generally, the weight for an observation that is \( i \) days old is equal to:

\[
w(i) = \frac{\lambda^{i-1} (1 - \lambda)}{1 - \lambda^n} \]

The implication of the age-weighted simulation is to reduce the impact of ghost effects and older events that may not reoccur. Note that this more general weighting scheme suggests that historical simulation is a special case where \( \lambda = 1 \) (i.e., no decay) over the estimation window.

**Professor’s Note:** You may recall from the Part I curriculum that this approach is also known as the hybrid approach.

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Volatility-weighted Historical Simulation

Another approach is to weight the individual observations by volatility rather than proximity to the current date. This was introduced by Hull and White to incorporate changing volatility in risk estimation.\(^2\) The intuition is that if recent volatility has increased, then using historical data will underestimate the current risk level. Similarly, if current volatility is markedly reduced, the impact of older data with higher periods of volatility will overstate the current risk level.

This process is captured in the expression below for estimating \(\text{VaR}\) on day \(T\). The expression is achieved by adjusting each daily return, \(r_{t,i}\) on day \(t\) upward or downward based on the then-current volatility estimate, \(\sigma_{t,i}\) (estimated from a GARCH or EWMA model) relative to the current volatility forecast on day \(T\).

\[
\begin{align*}
    r_{t,i}^* &= \left( \frac{\sigma_{T,i}}{\sigma_{t,i}} \right) r_{t,i} \\
    \text{where:} \\
    r_{t,i} &= \text{actual return for asset } i \text{ on day } t \\
    \sigma_{t,i} &= \text{volatility forecast for asset } i \text{ on day } t \text{ (made at the end of day } t-1) \\
    \sigma_{T,i} &= \text{current forecast of volatility for asset } i
\end{align*}
\]

Thus, the volatility-adjusted return, \(r_{t,i}^*\), is replaced with a larger (smaller) expression if current volatility exceeds (is below) historical volatility on day \(t\). Now, \(\text{VaR}\), \(\text{ES}\), and any other coherent risk measure can be calculated in the usual way after substituting historical returns with volatility-adjusted returns.

There are several advantages of the volatility-weighted method. First, it explicitly incorporates volatility into the estimation procedure in contrast to other historical methods. Second, the near-term \(\text{VaR}\) estimates are likely to be more sensible in light of current market conditions. Third, the volatility-adjusted returns allow for \(\text{VaR}\) estimates that are higher than estimates with the historical data set.

Correlation-weighted Historical Simulation

As the name suggests, this methodology incorporates updated correlations between asset pairs. This procedure is more complicated than the volatility-weighting approach, but it follows the same basic principles. Since the AIM does not require calculations, the exact matrix algebra would only complicate our discussion. Intuitively, the historical correlation (or equivalently variance-covariance) matrix needs to be adjusted to the new information environment. This is accomplished, loosely speaking, by “multiplying” the historic returns by the revised correlation matrix to yield updated correlation-adjusted returns.

Let us look at the variance-covariance matrix more closely. In particular, we are concerned with diagonal elements and the off-diagonal elements. The off-diagonal elements represent the current covariance between asset pairs. On the other hand, the diagonal elements represent the updated variances (covariance of the asset return with itself) of the individual assets.

\[
\Sigma = \begin{pmatrix}
\sigma_{i,i} & \sigma_{i,j} \\
\sigma_{j,i} & \sigma_{j,j}
\end{pmatrix} = \begin{pmatrix}
\text{Variance}(X_i) & \text{Cov}(X_i, X_j) \\
\text{Cov}(X_j, X_i) & \text{Variance}(X_j)
\end{pmatrix}
\]

Notice that updated variances were utilized in the previous approach as well. Thus, correlation-weighted simulation is an even richer analytical tool than volatility-weighted simulation because it allows for updated variances (volatilities) as well as covariances (correlations).

**Filtered Historical Simulation**

The filtered historical simulation is the most comprehensive, and hence most complicated, of the non-parametric estimators. The process combines the historical simulation model with conditional volatility models (like GARCH or asymmetric GARCH). Thus, the method contains both the attractions of the traditional historical simulation approach with the sophistication of models that incorporate changing volatility. In simplified terms, the model is flexible enough to capture conditional volatility and volatility clustering as well as a surprise factor that could have an asymmetric effect on volatility.

The model will forecast volatility for each day in the sample period and the volatility will be standardized by dividing by realized returns. Bootstrapping is used to simulate returns which incorporate the current volatility level. Finally, the VaR is identified from the simulated distribution. The methodology can be extended over longer holding periods or for multi-asset portfolios.

In sum, the filtered historical simulation method uses bootstrapping and combines the traditional historical simulation approach with rich volatility modeling. The results are then sensitive to changing market conditions and can predict losses outside the historical range. From a computational standpoint, this method is very reasonable even for large portfolios, and empirical evidence supports its predictive ability.

**Advantages and Disadvantages of Non-Parametric Methods**

AIM 2.4: Discuss the advantages and disadvantages of non-parametric estimation methods.

Any risk manager should be prepared to use non-parametric estimation techniques. There are some clear advantages to non-parametric methods, but there is some danger as well. Therefore, it is incumbent to understand the advantages, the disadvantages, and the appropriateness of the methodology for analysis.
Advantages of non-parametric methods include the following:

- Intuitive and often computationally simple (even on a spreadsheet).
- Not hindered by parametric violations of skewness, fat-tails, et cetera.
- Avoids complex variance-covariance matrices and dimension problems.
- Data is often readily available and does not require adjustments (e.g., financial statements adjustments).
- Can accommodate more complex analysis (e.g., by incorporating age-weighting with volatility-weighting).

Disadvantages of non-parametric methods include the following:

- Analysis depends critically on historical data.
- Volatile data periods lead to VaR and ES estimates that are too high.
- Quiet data periods lead to VaR and ES estimates that are too low.
- Difficult to detect structural shifts/regime changes in the data.
- Cannot accommodate plausible large impact events if they did not occur within the sample period.
- Difficult to estimate losses significantly larger than the maximum loss within the data set (historical simulation cannot; volatility-weighting can, to some degree).
- Need sufficient data, which may not be possible for new instruments or markets.
Cross Reference to GARP Assigned Reading – Dowd, Chapter 4

Key Concepts

1. Bootstrapping involves resampling a subset of the original data set with replacement. Each draw (subsample) yields a coherent risk measure (VaR or ES). The average of the risk measures across all samples is then the best estimate.

2. The discreteness of historical data reduces the number of possible VaR estimates since historical simulation cannot adjust for significance levels between ordered observations. However, non-parametric density estimation allows the original histogram to be modified to fill in these gaps. The process connects the midpoints between successive columns in the histogram. The area is then “removed” from the upper bar and “placed” in the lower bar, which creates a “smooth” function between the original data points.

3. One important limitation to the historical simulation method is the equal-weight assumed for all data in the estimation period, and zero weight otherwise. This arbitrary methodology can be improved by using age-weighted simulation, volatility-weighted simulation, correlation-weighted simulation, and filtered historical simulation.

4. The age-weighted simulation method adjusts the most recent (distant) observations to be more (less) heavily weighted.

5. The volatility-weighting procedure incorporates the possibility that volatility may change over the estimation period, which may understate or overstate current risk by including stale data. The procedure replaces historic returns with volatility-adjusted returns; however, the actual procedure of estimating VaR is unchanged (i.e., only the data inputs change).

6. Correlation-weighted simulation updates the variance-covariance matrix between the assets in the portfolio. The off-diagonal elements represent the covariance pairs while the diagonal elements update the individual variance estimates. Therefore, the correlation-weighted methodology is more general than the volatility-weighting procedure by incorporating both variance and covariance adjustments.

7. Filtered historical simulation is the most complex estimation method. The procedure relies on bootstrapping of standardized returns based on volatility forecasts. The volatility forecasts arise from GARCH or similar models and are able to capture conditional volatility, volatility clustering, and/or asymmetry.

8. Advantages of non-parametric models include: data can be skewed or have fat tails; they are conceptually straightforward; there is readily available data; and they can accommodate more complex analysis. Disadvantages focus mainly on the use of historical data, which limits the VaR forecast to (approximately) the maximum loss in the data set; they are slow to respond to changing market conditions; they are affected by volatile (quiet) data periods; and they cannot accommodate plausible large losses if not in the data set.
1. Johanna Roberto has collected a data set of 1,000 daily observations on equity returns. She is concerned about the appropriateness of using parametric techniques as the data appears skewed. Ultimately, she decides to use historical simulation and bootstrapping to estimate the 5% VaR. Which of the following steps is most likely to be part of the estimation procedure?
   A. Filter the data to remove the obvious outliers.
   B. Repeated sampling with replacement.
   C. Identify the tail region from reordering the original data.
   D. Apply a weighting procedure to reduce the impact of older data.

2. All of the following approaches improve the traditional historical simulation approach for estimating VaR except the:
   A. volatility-weighted historical simulation.
   B. age-weighted historical simulation.
   C. market-weighted historical simulation.
   D. correlation-weighted historical simulation.

3. Which of the following statements about age-weighting is most accurate?
   A. The age-weighting procedure incorporates estimates from GARCH models.
   B. If the decay factor in the model is close to 1, there is persistence within the data set.
   C. When using this approach, the weight assigned on day $i$ is equal to:
     \[ w(i) = \lambda^{i-1} \times (1 - \lambda) / (1 - \lambda^i) . \]
   D. The number of observations should at least exceed 250.

4. Which of the following statements about volatility-weighting is true?
   A. Historic returns are adjusted, and the VaR calculation is more complicated.
   B. Historic returns are adjusted, and the VaR calculation procedure is the same.
   C. Current period returns are adjusted, and the VaR calculation is more complicated.
   D. Current period returns are adjusted, and the VaR calculation is the same.

5. All of the following items are generally considered advantages of non-parametric estimation methods except:
   A. ability to accommodate skewed data.
   B. availability of data.
   C. use of historical data.
   D. little or no reliance on covariance matrices.
1. B Bootstrapping from historical simulation involves repeated sampling with replacement. The 5% VaR is recorded from each sample draw. The average of the VaRs from all the draws is the VaR estimate. The bootstrapping procedure does not involve filtering the data or weighting observations. Note that the VaR from the original data set is not used in the analysis.

2. C Market-weighted historical simulation is not discussed in this topic. Age-weighted historical simulation weights observations higher when they appear closer to the event date. Volatility-weighted historical simulation adjusts for changing volatility levels in the data. Correlation-weighted historical simulation incorporates anticipated changes in correlation between assets in the portfolio.

3. B If the intensity parameter (i.e., decay factor) is close to 1, there will be persistence (i.e., slow decay) in the estimate. The expression for the weight on day i has i in the exponent when it should be n. While a large sample size is generally preferred, some of the data may no longer be representative in a large sample.

4. B The volatility-weighting method adjusts historic returns for current volatility. Specifically, return at time t is multiplied by (current volatility estimate / volatility estimate at time t). However, the actual procedure for calculating VaR using a historical simulation method is unchanged; it is only the inputted data that changes.

5. C The use of historical data in non-parametric analysis is a disadvantage, not an advantage. If the estimation period was quiet (volatile) then the estimated risk measures may understate (overstate) the current risk level. Generally, the largest VaR cannot exceed the largest loss in the historical period. On the other hand, the remaining choices are all considered advantages of non-parametric methods. For instance, the non-parametric nature of the analysis can accommodate skewed data, data points are readily available, and there is no requirement for estimates of covariance matrices.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**MODELING DEPENDENCE: CORRELATIONS AND COPULAS**

**ExAm Focus**

A copula is a joint multivariate distribution that describes how variables from marginal distributions come together. Copulas provide an alternative measure of dependence between random variables that is not subject to the same limitations as correlation in applications as a risk measurement. When the variables from marginal distributions are continuous, copulas can be used to estimate tail dependence by defining a coefficient of (upper) tail dependence of $X$ and $Y$. Thus, copulas provide an asymptotic measure of dependence for extreme events that are oftentimes related. The material in this topic is relatively complex, so your focus here should be on gaining a general understanding of what a copula function is used for.

**Measuring Dependence with Correlation**

AIM 3.1: Explain the drawbacks of using correlation to measure dependence.

Correlation is a good measure of dependence for random variables that are normally distributed. However, correlation is not an adequate measure of dependence for all elliptical distributions. If returns have a multivariate normal distribution, then a zero correlation implies that returns are independent. Conversely, if returns are not normally distributed, then a zero correlation does not necessarily imply returns are independent. Another drawback of using correlation to measure dependence is that it is not invariant to transformations. For example, the correlation of two random variables $X$ and $Y$ will not be the same as the correlation of $\ln(X)$ and $\ln(Y)$.

As mentioned, correlation is not a good measure of dependence when the underlying variables have distributions that are non-elliptical. Problems arise for distributions that either have heavy-tails with infinite variance or for trended return series that are not cointegrated (i.e., linear combinations of two or more time series are not stationary). This is problematic because correlation can only be defined when variances are finite.

Even when variances are defined, correlation will give a misleading indication of dependence for non-elliptical distributions. For example, consider a random variable $X$ that is normally distributed with a mean of 0 and a standard deviation of 1. Now transform the variables such that $Z_1 = e^X$, $Z_2 = e^{2X}$, and $Z_3 = e^{-X}$. $Z_1$ and $Z_2$ will move perfectly with one another as they are *comonotonic* (i.e., always move in a similar direction). The estimated correlation between $Z_1$ and $Z_2$ is 0.731. $Z_1$ and $Z_3$ will move perfectly in the opposite direction with one another as they are *countemonotonic*. The estimated correlation between $Z_1$ and $Z_3$ is $-0.367$. Thus, when distributions are not elliptical the correlations may not vary between $-1$ and 1, giving a misleading indication of dependency. Correlation and marginal distributions are not sufficient in describing the joint multivariate distribution. Therefore,
correlation will not give us adequate information regarding dependency, especially in the tails of non-elliptical distributions.

**Measuring Dependence with Copulas**

**AIM 3.2:** Describe how copulas provide an alternative measure of dependence.

Copulas provide an alternative measure of dependence between random variables. A copula is defined as a function that joins a multivariate distribution function to a collection of univariate marginal distribution functions. In other words, the marginal distributions describe the probability of outcomes of variables on their own, and the copula is the joint multivariate distribution that describes the probability of outcomes as variables come together. A copula extracts the dependence structure from the joint distribution function.

For example, assume there are two random variables $X$ and $Y$. $F(x,y)$ is a joint distribution function with continuous marginal function $F_x(x) = u$ for the $X$ variable and continuous marginal function $F_y(y) = v$ for the $Y$ variable. The copula is written as a unique function, $C(u,v)$, as follows:

$$F(x,y) = C(u,v)$$

The copula describes the dependence structure of the joint distribution function $F(x,y)$.

After specifying the marginal distribution functions and the copula that represents their dependence structure, the joint distribution function can be modeled by estimating parameters involved and applying the copula function to the marginal distributions. The copulas are then used as an alternative risk measure that is not dependent on restrictive assumptions of the underlying variable distributions.

**Common Copula Functions**

**AIM 3.3:** Identify basic examples of copulas.

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*Professor’s Note: In this AIM, we identify common copulas and their equations. For the exam, you only need a general understanding of the types of copulas in existence. You will most likely not need to know these complex equations for the exam.*

The independence, minimum, and maximum copulas shown below represent the simplest forms of copula equations. The independent copula is typically used where random variables $X$ and $Y$ are independent. The minimum copula is used where $X$ and $Y$ are positively dependent or comonotonic as they always move in the same direction. The maximum copula is used where $X$ and $Y$ are negatively correlated or countermonotonic as they always move in opposite directions.

- Independence: $C_{\text{ind}}(u,v) = uv$
- Minimum: $C_{\text{min}}(u,v) = \min[u,v]$
- Maximum: $C_{\text{max}}(u,v) = \max[u + v - 1, 0]$
The Gaussian (or normal) copula depends only on the correlation coefficient, $\rho$, where $-1 \leq \rho \leq 1$. The $\phi$ symbol is the univariate standard normal distribution function. The $t$-copula is the generalized form of the Gaussian copula.

**Gaussian:**

$$C_{\rho}^G(x,y) = \frac{1}{2\pi(1-\rho^2)^{0.5}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{(s^2 - 2\rho st + t^2)}{2(1-\rho^2)}\right] ds dt$$

**$t$-copula:**

$$C_{\rho}^t(x,y) = \frac{1}{2(1-\rho^2)^{0.5}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{(s^2 - 2\rho st + t^2)}{v(1-\rho^2)}\right] ds dt$$

*Professor’s Note: As will be shown in the credit risk material in Book 2, one-factor Gaussian copula models are used to price tranches of collateralized debt obligations (CDOs).*

Another common copula is the Gumbel (or logistic) copula. The dependence between two variables is determined by $\beta$, where $0 < \beta \leq 1$. The Gumbel copula is used to model multivariate extremes, where the Gaussian copula is not appropriate.

**Gumbel:**

$$C_{\beta}^G(x,y) = \exp\left[-\left\{\left(-\log x\right)^{1/\beta} + \left(-\log y\right)^{1/\beta}\right\}\right]$$

**Archimedean copulas** are a group of copulas that utilize a generator function, $\varphi$. This generator function is continuously decreasing and convex such that $\varphi(1) = 0$. Another group of copulas is known as the extreme value (EV) copulas. An EV copula is appropriate when the joint multivariate function is a multivariate generalized extreme value (GEV) distribution. Other examples of EV copulas are the Gumbel II and Galambos, where the $\beta$ ranges are defined as $[0,1]$ and $[0,\infty)$, respectively.

**Archimedean:**

$$C(u,v) = \varphi^{-1}[\varphi(u) + \varphi(v)]$$

**Extreme value (EV):**

$$C(u',v') = [C(u,v)]^\beta$$

**Gumbel II:**

$$uv \exp\left[\frac{\beta(uv)}{u + v}\right]$$

**Galambos:**

$$uv \exp\left[\left(u^{-\beta} + v^{-\beta}\right)^{-1/\beta}\right]$$
Tail Dependence

AIM 3.4: Explain how tail dependence can be investigated using copulas.

Extreme events are oftentimes related and an asymptotic measure of the dependence of these events is known as tail dependence. When marginal distributions are continuous, copulas are used to estimate tail dependence by defining a coefficient of tail dependence, \( \lambda \). The following conditional probability expression solves for tail dependence assuming that \( u \) approaches one.

\[
\lambda = \mathbb{P}(Y > F^{-1}_Y(u) | X > F^{-1}_X(u))
\]

For example, in the Gumbel copula, it can be shown that \( \lambda = 2 - 2^\beta \). Thus, \( X \) and \( Y \) are asymptotically dependent as long as \( \beta < 1 \). Similarly, in the Gaussian copula, \( X \) and \( Y \) are asymptotically dependent as long as \( \rho < 1 \). When these conditions hold, extreme events will be independent as they occur further out in the tail.

If the correlation coefficient is greater than \(-1\) in a \( t \)-distribution, then \( X \) and \( Y \) will be asymptotically dependent. Conversely, in a normal distribution, extreme events will not be asymptotically dependent because the tails of the normal distribution are not as heavy as the \( t \)-distribution.
1. When returns are not normally distributed, a zero correlation does not imply returns are independent and correlation is not invariant to transformations. In addition, correlation is not a good measure of dependence when the underlying variables have distributions that are non-elliptical, because correlation can only be defined when variances are finite.

2. A copula is defined as a function that joins a multivariate distribution function to a collection of univariate marginal distribution functions. They are used as a risk measure that is not dependent on restrictive assumptions of the underlying variable distributions.

3. When marginal distributions are continuous, copulas are used to estimate tail dependence by defining a coefficient of (upper) tail dependence.
1. Which of the following statements regarding correlation dependence is false?
   A. Correlation is a good measure of dependence for random variables that are normally distributed.
   B. Correlation is not an adequate measure of dependence for all elliptical distributions.
   C. If return distributions are non-elliptical, then a zero correlation implies returns are independent.
   D. A drawback of using correlation to measure dependence is that correlation is not invariant to transformations.

2. Which of the following statements regarding common copula functions is incorrect?
   A. The minimum copula is used where random variables are comonotonic.
   B. The maximum copula is used where random variables are countermonotonic.
   C. The Gaussian copula depends only on the correlation coefficient, \( \rho \), where \(-1 \leq \rho \leq 1\).
   D. Archimedean copulas are a group of copulas whose distribution function is strictly decreasing and concave.

3. A generalized form of the Gaussian copula is known as a(n):
   A. Archimedean copula.
   B. extreme value copula.
   C. Gumbel II copula.
   D. \( t \)-copula.

4. Extreme events are oftentimes related, and an asymptotic measure of the dependence of these events is known as:
   A. correlation dependence.
   B. extreme value dependence.
   C. Gaussian dependence.
   D. tail dependence.

5. The risk measure that extracts the dependence structure from the joint distribution function created from several continuous marginal distribution functions is known as a:
   A. copula.
   B. volatility.
   C. correlation.
   D. multivariate correlation.
1. C If returns have a multivariate normal distribution, then a zero correlation implies that risks are independent. Conversely, if returns are not normally distributed, then a zero correlation does not imply returns are independent.

2. D Archimedean copulas are a group of copulas such that the joint distribution is represented by $C(u,v) = \psi^{-1}[\psi(u) + \psi(v)]$ where the generator function is strictly decreasing and convex.

3. D The $t$-copula is a generalized form of the Gaussian copula.

4. D Extreme events are oftentimes related, and an asymptotic measure of the dependence of these events is known as tail dependence.

5. A A copula is a risk measure that extracts the dependence structure from the joint distribution function created from several continuous marginal distribution functions.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**PARAMETRIC APPROACHES (II): EXTREME VALUE**

**Exam Focus**

Extreme values are important for risk management because they are associated with catastrophic events such as the failure of large institutions and market crashes. Since they are rare, modeling such events is a challenging task. In this topic, we will address the generalized extreme value (GEV) distribution, and the peaks-over-threshold approach, as well as discuss how peaks-over-threshold converges to the generalized Pareto distribution. Also of some importance is the calculation of VaR and expected shortfall assuming peaks-over-threshold estimates.

**Managing Extreme Values**

AIM 4.1: Explain the importance and challenges of extreme values for risk management.

The occurrence of extreme events is rare; however, it is crucial to identify these extreme events for risk management since they can prove to be very costly. Extreme values are the result of large market declines or crashes, the failure of major institutions, the outbreak of financial or political crises, or natural catastrophes. The challenge of analyzing and modeling extreme values is that there are only a few observations for which to build a model, and there are ranges of extreme values that have yet to occur.

To meet the challenge, researchers must assume a certain distribution. The assumed distribution will probably not be identical to the true distribution; therefore, some degree of error will be present. Researchers usually choose distributions based on measures of central tendency, which misses the issue of trying to incorporate extreme values. Researchers need approaches that specifically deal with extreme value estimation. Incidentally, researchers in many fields other than finance face similar problems. In flood control, for example, analysts have to model the highest possible flood line when building a dam, and this estimation would most likely require a height above observed levels of flooding to date.

**Extreme Value Theory**

AIM 4.2: Describe extreme value theory (EVT) and its use in risk management.

Extreme value theory (EVT) is a branch of applied statistics that has been developed to address problems associated with extreme outcomes. EVT focuses on the unique aspects of extreme values and is different from “central tendency” statistics, in which the central-limit
theorem plays an important role. Extreme value theorems provide a template for estimating the parameters used to describe extreme movements.

One approach for estimating parameters is the Fisher–Tippett theorem (1928). According to this theorem, as the sample size \( n \) gets large, the distribution of extremes, denoted \( M_n \), converges to the following distribution known as the generalized extreme value (GEV) distribution:

\[
F(X | \xi, \mu, \sigma) = \exp \left[ -\left( 1 + \xi \frac{X - \mu}{\sigma} \right)^{-1/\xi} \right] \text{ if } \xi \neq 0
\]

\[
F(X | \xi, \mu, \sigma) = -\exp \left( \frac{X - \mu}{\sigma} \right) \text{ if } \xi = 0
\]

For these formulas, the following restriction holds for random variable \( X \):

\[
1 + \xi \frac{X - \mu}{\sigma} > 0
\]

The parameters \( \mu \) and \( \sigma \) are the location parameter and scale parameter, respectively, of the limiting distribution. Although related to the mean and variance, they are not the same. The symbol \( \xi \) is the tail index and indicates the shape (or heaviness) of the tail of the limiting distribution. There are three general cases of the GEV distribution:

1. \( \xi > 0 \), the GEV becomes a Frechet distribution, and the tails are “heavy” as is the case for the t-distribution and Pareto distributions.

2. \( \xi = 0 \), the GEV becomes the Gumbel distribution, and the tails are “light” as is the case for the normal and log-normal distributions.

3. \( \xi < 0 \), the GEV becomes the Weibull distribution, and the tails are “lighter” than a normal distribution.

Distributions where \( \xi < 0 \) do not often appear in financial models; therefore, financial risk management analysis can essentially focus on the first two cases: \( \xi > 0 \) and \( \xi = 0 \). Therefore, one practical consideration the researcher faces is whether to assume either \( \xi > 0 \) or \( \xi = 0 \) and apply the respective Frechet or Gumbel distributions and their corresponding estimation procedures. There are three basic ways of making this choice.

1. The researcher is confident of the parent distribution. If the researcher is confident it is a t-distribution, for example, then the researcher should assume \( \xi > 0 \).

2. The researcher applies a statistical test and cannot reject the hypothesis \( \xi = 0 \). In this case, the researcher uses the assumption \( \xi = 0 \).

3. The researcher may wish to be conservative and assume \( \xi > 0 \) to avoid model risk.
**Peaks-Over-Threshold**

**AIM 4.3:** Describe the peaks-over-threshold (POT) approach.

The peaks-over-threshold (POT) approach is an application of extreme value theory to the distribution of excess losses over a high threshold. The POT approach generally requires fewer parameters than approaches based on extreme value theorems. The POT approach provides the natural way to model values that are greater than a high threshold, and in this way, it corresponds to the GEV theory by modeling the maxima or minima of a large sample.

The POT approach begins by defining a random variable \( X \) to be the loss. We define \( u \) as the threshold value for positive values of \( x \), and the distribution of excess losses over our threshold \( u \) as:

\[
F_u(x) = P[X - u \leq x | X > u] = \frac{F(x + u) - F(u)}{1 - F(u)}
\]

This is the conditional distribution for \( X \) given that the threshold is exceeded by no more than \( x \). The parent distribution of \( X \) can be normal or lognormal, however, it will usually be unknown.

**Generalized Pareto Distribution**

**AIM 4.5:** Describe the parameters of a generalized Pareto (GP) distribution.

The Gnedenko-Pickands-Balkema-deHaan (GPBdH) theorem says that as \( u \) gets large, the distribution \( F_u(x) \) converges to a generalized Pareto distribution (GPD), such that:

\[
1 - \left[ 1 + \frac{\xi x}{\beta} \right]^{-1/\xi} \quad \text{if } \xi \neq 0
\]

\[
1 - \exp\left( -\frac{x}{\beta} \right) \quad \text{if } \xi = 0
\]

The distribution is defined for the following regions:

- \( x \geq 0 \) for \( \xi \geq 0 \) and \( 0 \leq x \leq -\beta/\xi \) for \( \xi < 0 \)

The tail (or shape) index parameter, \( \xi \), is the same as it is in GEV theory. It can be positive, zero, or negative, but we are mainly interested in the cases when it is zero or positive. Here, the beta symbol, \( \beta \), represents the scale parameter.

The GPD exhibits a curve that dips below the normal distribution prior to the tail. It then moves above the normal distribution until it reaches the extreme tail. The GPD then provides a linear approximation of the tail, which more closely matches empirical data.
AIM 4.6: Explain the tradeoffs in setting the threshold level when applying the GP distribution.

Since all distributions of excess losses converge to the GPD, it is the natural model for excess losses. It requires a selection of $u$, which determines the number of observations, $N_u$, in excess of the threshold value. Choosing the threshold involves a tradeoff. It needs to be high enough so the GPD theory can apply, but it must be low enough so that there will be enough observations to apply estimation techniques to the parameters.

**VAR AND EXPECTED SHORTFALL**

AIM 4.7: Compute VaR and expected shortfall using the POT approach, given various parameter values.

One of the goals of using the POT approach is to ultimately compute the value at risk (VaR). From estimates of VaR, we can derive the expected shortfall (a.k.a. conditional VaR). Expected shortfall is viewed as an average or expected value of all losses greater than the VaR. An expression for this is: $E[L_p \mid L_p > VaR]$. Because it gives an insight into the distribution of the size of losses greater than the VaR, it has become a popular measure to report along with VaR.

The expression for VaR using POT parameters is given as follows:

$$VaR = u + \frac{\beta}{\xi} \left[ \frac{n}{N_u} (1 - \text{confidence level}) \right]^{-\xi} - 1$$

where:
- $u$ = threshold (in percentage terms)
- $n$ = number of observations
- $N_u$ = number of observations that exceed threshold

The expected shortfall can then be defined as:

$$ES = \frac{VaR \cdot \beta - \xi u}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi}$$
Example: Compute VaR and expected shortfall given POT estimates

Assume the following observed parameter values:

- $\beta = 0.75$.
- $\xi = 0.25$.
- $u = 1\%$.
- $N_u/n = 5\%$.

Compute the 1% VaR in percentage terms and the corresponding expected shortfall measure.

**Answer:**

\[
VaR = 1 + \frac{0.75}{0.25} \left[ \frac{1}{0.05} \left( 1 - 0.99 \right) \right]^{-0.25} - 1 = 2.486\%
\]

\[
ES = \frac{2.486}{1 - 0.25} + \frac{0.75 - 0.25 \times 1}{1 - 0.25} = 3.981\%
\]

**Generalized Extreme Value and Peaks-Over-Threshold**

**AIM 4.4:** Compare generalized extreme value and POT.

Extreme value theory is the source of both the GEV and POT approaches. These approaches are similar in that they both have a tail parameter denoted $\xi$. There is a subtle difference in that GEV theory focuses on the distributions of extremes, whereas POT focuses on the distribution of values that exceed a certain threshold. Although very similar in concept, there are cases where a researcher might choose one over the other. Here are three considerations.

1. GEV requires the estimation of one more parameter than POT. The most popular approaches of the GEV can lead to loss of useful data relative to the POT.

2. The POT approach requires a choice of a threshold, which can introduce additional uncertainty.

3. The nature of the data may make one preferable to the other.
Multivariate EVT

AIM 4.8: Explain the importance of multivariate EVT for risk management.

Multivariate EVT is important because we can easily see how extreme values can be dependant on each other. A terrorist attack on oil fields will produce losses for oil companies, but it is likely that the value of most financial assets will also be affected. We can imagine similar relationships between the occurrence of a natural disaster and a decline in financial markets as well as markets for real goods and services.

Multivariate EVT has the same goal as univariate EVT in that the objective is to move from the familiar central-value distributions to methods that estimate extreme events. The added feature is to apply the EVT to more than one random variable at the same time. This introduces the concept of tail dependence, which is the central focus of multivariate EVT. Assumptions of an elliptical distribution and the use of a covariance matrix are of limited use for multivariate EVT.

Modeling multivariate extremes requires the use of copulas as was examined in the previous topic. Multivariate EVT says that the limiting distribution of multivariate extreme values will be a member of the family of EV copulas, and we can model multivariate EV dependence by assuming one of these EV copulas. The copulas can also have as many dimensions as appropriate and congruous with the number of random variables under consideration. However, the increase in the dimensions will present problems. If a researcher has two independent variables and classifies univariate extreme events as those that occur one time in a 100, this means that the researcher should expect to see one multivariate extreme event (i.e., both variables taking extreme values) only one time in \(100 \times 100 = 10,000\) observations. For a trinomial distribution, that number increases to \(1,000,000\). This reduces drastically the number of multivariate extreme observations to work with, and increases the number of parameters to estimate.
1. Estimating extreme values is important since they can be very costly. The challenge is that since they are rare, many have not even been observed. Thus, it is difficult to model them.

2. Extreme value theory (EVT) can be used to model extreme events in financial markets and to compute VaR, as well as expected shortfall.

3. The peaks-over-threshold (POT) approach is an application of extreme value theory. It models the values that occur over a given threshold. It assumes that observations beyond the threshold follow a generalized Pareto distribution whose parameters can be estimated.

4. The GEV and POT approach have the same goal and are built on the same general principles of extreme value theory. They even share the same shape parameter: \( \xi \).

5. The parameters of a generalized Pareto distribution (GPD) are the scale parameter \( \beta \) and the shape parameter \( \xi \). Both of these can be estimated using maximum-likelihood techniques.

6. When applying the generalized Pareto distribution, the researcher must choose a threshold. There is a tradeoff because the threshold must be high enough so that the GPD applies, but it must be low enough so that there are sufficient observations above the threshold to estimate the parameters.

7. For the POT approach, the VaR and expected shortfall formulas are as follows:

\[
\text{VaR} = u + \beta \left[ \frac{n}{N_u} (1 - \text{confidence level}) \right]^{-\xi} - 1
\]

\[
\text{ES} = \frac{\text{VaR}}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi}
\]

8. Multivariate EVT is important because many extreme values are dependent on each other, and elliptical distribution analysis and correlations are not useful in the modeling of extreme values for multivariate distributions. Modeling multivariate extremes requires the use of copulas. Given that more than one random variable is involved, modeling these extremes can be even more challenging because of the rarity of multiple extreme values occurring at the same time.
1. According to the Fisher-Tippett theorem, as the sample size $n$ gets large, the distribution of extremes converges to:
   A. a normal distribution.
   B. a uniform distribution.
   C. a generalized Pareto distribution.
   D. a generalized extreme value distribution.

2. The peaks-over-threshold approach generally requires:
   A. more estimated parameters than the GEV approach and shares one parameter with the GEV.
   B. fewer estimated parameters than the GEV approach and shares one parameter with the GEV.
   C. more estimated parameters than the GEV approach and does not share any parameters with the GEV approach.
   D. fewer estimated parameters than the GEV approach and does not share any parameters with the GEV approach.

3. In setting the threshold in the POT approach, which of the following statements is the most accurate? Setting the threshold relatively high makes the model:
   A. more applicable but decreases the number of observations in the modeling procedure.
   B. less applicable and decreases the number of observations in the modeling procedure.
   C. more applicable but increases the number of observations in the modeling procedure.
   D. less applicable but increases the number of observations in the modeling procedure.

4. A researcher using the POT approach observes the following parameter values: $\beta = 0.9$, $\xi = 0.15$, $u = 2\%$ and $N_x/n = 4\%$. The 5% VaR in percentage terms is:
   A. 1.034.
   B. 1.802.
   C. 2.204.
   D. 16.559.

5. Given a VaR equal to 2.56, a threshold of 1%, a shape parameter equal to 0.2, and a scale parameter equal to 0.3, what is the expected shortfall?
   A. 3.325.
   B. 3.526.
   C. 3.777.
   D. 4.086.
The Fisher-Tippett theorem says that as the sample size $n$ gets large, the distribution of extremes, denoted $M_n$, converges to a generalized extreme value (GEV) distribution.

The POT approach generally has fewer parameters, but both POT and GEV approaches share the tail parameter $\xi$.

There is a trade-off in setting the threshold. It must be high enough for the appropriate theorems to hold, but if set too high, there will not be enough observations to estimate the parameters.

$$\text{VaR} = 2 + \frac{0.9}{0.15} \left[ \frac{1}{0.04} (1 - 0.95) \right]^{-0.15} - 1$$

$$\text{VaR} = 1.802$$

$$\text{ES} = \frac{\text{VaR}}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi} = \frac{2.560}{1 - 0.2} + \frac{0.3 - 0.2 \times 1}{1 - 0.2} = 3.325$$
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**BACKTESTING VaR**

**Exam Focus**

We use value at risk (VaR) methodologies to model risk. With VaR models, we seek to approximate the changes in value that our portfolio would experience in response to changes in the underlying risk factors. Model validation incorporates several methods that we use in order to determine how close our approximations are to actual changes in value. Through model validation, we are able to determine what confidence to place in our models, and we have the opportunity to improve their accuracy. Validation can be done through backtesting (also called a reality check), stress testing, and with independent review and oversight.

**Backtesting**

AIM 5.1: Define backtesting and exceptions and explain the importance of backtesting VaR models.

Backtesting is the process of comparing losses predicted by the value at risk (VaR) model to those actually experienced over the testing period. If a model were completely accurate, we would expect VaR loss limits to be exceeded (this is called an exception) with the same frequency predicted by the confidence level used in the VaR model.

For example, if a VaR of $10 million is calculated at a 95% confidence level, we expect to have exceptions (losses exceeding $10 million) 5% of the time. If exceptions are occurring with greater frequency, we may be underestimating the actual risk. If exceptions are occurring less frequently, we may be overestimating risk and misallocating capital as a result.

VaR models are based on static portfolios, while actual portfolio compositions are constantly changing as relative prices change and positions are bought and sold. These risk factors affect actual profit and loss, but they are not included in the VaR model. We can minimize such effects by backtesting with a daily holding period, but the modeled returns are comparable to the hypothetical return that would be experienced had the portfolio remained constant for the holding period. Generally, we compare the VaR model returns to cleaned returns (actual returns adjusted for all changes that arise from changes that are not mark to market, like funding costs and fee income).
**Backtesting Exceptions**

AIM 5.2: Explain the significant difficulties in backtesting a VaR model.

AIM 5.3: Explain the framework of backtesting models with the use of exceptions or failure rates.

AIM 5.4: Define and identify type I and type II errors.

The very design of VaR models includes the use of confidence levels. We expect to have a frequency of exceptions that corresponds to the confidence level used for the model. If we use a 95% confidence level, we expect to find exceptions in 5% of the instances. The backtesting period constitutes a limited sample, and we would not expect to find the predicted number of exceptions in every sample. How do we determine if the actual number of exceptions is acceptable? If we expect five exceptions and find eight, is that too many? What about nine? At some level, we must reject the model, and we need to know that level.

**Using Failure Rates in Model Verification**

An unbiased measure of the number of exceptions as a proportion of the number of samples is called the failure rate. The probability of exception equals one minus the confidence level \( p = 1 - c \). If we use \( N \) to represent the number of exceptions and \( T \) to represent the number of samples, then \( N/T \) is the failure rate. A sample cannot be used to determine with absolute certainty whether or not the model is accurate. However, we can determine the accuracy of the model and the probability of having the number of exceptions that we experienced. By determining a range for the number of exceptions that we would accept, we must strike a balance between the chances of rejecting an accurate model (Type I error) and the chances of accepting an inaccurate model (Type II error). We can establish such ranges at different confidence levels using a binomial probability distribution and the number of samples. The confidence level at which we choose to accept or reject a model is not related to the confidence level at which VaR was calculated. In evaluating the accuracy of the model, we are comparing the number of exceptions observed with the maximum number of exceptions that would be expected from a correct model at a given confidence level.

Kupiec (1995)\(^1\) determined a measure to accept or reject models using the tail points of a log-likelihood ratio:

\[
LR_{uc} = -2\ln\left(1 - p\right)^{T-N}p^N + 2\ln\left[1 - (N/T)\right]^{T-N}(N/T)^N
\]

where \( p, T, \) and \( N \) are as defined above, and \( LR_{uc} \) is the test statistic for unconditional coverage (uc).

We would reject the hypothesis that the model is correct if the \( LR > 3.84 \). In other words, this \( LR \) value is used to determine the range of acceptable exceptions without rejecting the VaR model at the 95% confidence level of the log-likelihood test.

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For instance, suppose we are backtesting a daily holding period VaR model that was constructed using a 97.5% confidence level over a 255-day period. If the model is accurate, the expected number of exceptions will be 2.5% of 255, or 6.375. We know that even if our model is precise, there will be some variation in the number of exceptions between samples. The mean of the samples will approach 6.375 as the number of samples increases. However, we also know that even if the model is incorrect, we might still end up with the number of exceptions at or near 6.375.

Figure 1 shows the calculated values of LR with 255 samples for a number of VaR confidence levels and a sample size of 255. The bold areas in Figure 1 correspond to LRs greater than 3.84. We would not reject the model if the number of exceptions in our sample is greater than 2 and less than 12, since these limits correspond to an LR < 3.84. In other words, this range of exceptions would not result in rejecting the model at a 97.5% confidence level.

<table>
<thead>
<tr>
<th>Confidence Level</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>97.5%</td>
<td>7.16</td>
<td>4.19</td>
<td>2.27</td>
<td>1.04</td>
<td>0.33</td>
<td>0.02</td>
<td>0.06</td>
<td>0.39</td>
<td>0.98</td>
<td>1.81</td>
<td>2.84</td>
<td>4.06</td>
</tr>
<tr>
<td>98.0%</td>
<td>5.01</td>
<td>2.49</td>
<td>1.03</td>
<td>0.26</td>
<td>0.00</td>
<td>0.15</td>
<td>0.65</td>
<td>1.44</td>
<td>2.48</td>
<td>3.76</td>
<td>5.25</td>
<td>6.93</td>
</tr>
<tr>
<td>99.0%</td>
<td>1.24</td>
<td>0.13</td>
<td>0.08</td>
<td>0.71</td>
<td>1.86</td>
<td>3.42</td>
<td>5.32</td>
<td>7.51</td>
<td>9.97</td>
<td>12.65</td>
<td>15.55</td>
<td>18.63</td>
</tr>
</tbody>
</table>

It is difficult to backtest VaR models constructed with higher levels of confidence, simply because the number of exceptions is often not high enough to provide meaningful information. As shown in Figure 1, with higher confidence levels, the range of acceptable exceptions is small. Thus, it becomes difficult to determine if the model is overstating risks (i.e., fewer than expected exceptions) or if the number of exceptions is simply at the lower range of acceptable.

**Using VaR to Measure Potential Losses**

Often the purpose of using VaR is to measure some level of potential losses. There are two theories about choosing a holding period for this application. The first theory is that the holding period should correspond to the amount of time required to either liquidate or hedge the portfolio. Thus, VaR would calculate possible losses before corrective action could take effect. The second theory is that the holding period should be chosen to match the period over which the portfolio is not expected to change due to nonrisk-related activity (e.g., trading). The two theories are not that different. For example, many banks use a daily VaR to correspond with the daily profit and loss measures. In this application, the holding period is more significant than the confidence level.
BASEL COMMITTEE RULES FOR BACKTESTING

AIM 5.6: Describe the Basel rules for backtesting.

In the backtesting process, we attempt to strike a balance between the probability of a Type I error and a Type II error. The Basel Committee requires that market VaR be calculated at the 99% confidence level and backtested over the past year. At the 99% confidence level, we would expect to have 2.5 exceptions (250 x 0.01) each year. In order to compensate for using inaccurate models, the committee has established a scale of the number of exceptions and corresponding increases in the capital multiplier, k. As you will see in the Basel material in Book 3, the multiplier is normally 3 but can be increased to as much as 4, based on the accuracy of the bank's VaR model. Figure 2 shows this scale. Increasing k significantly increases the amount of capital a bank must hold and lowers the bank's performance measures, like return on equity.

Figure 2: Basel Penalty Zones

<table>
<thead>
<tr>
<th>Zone</th>
<th>Number of Exceptions</th>
<th>Multiplier (k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Green</td>
<td>0 to 4</td>
<td>3.00</td>
</tr>
<tr>
<td>Yellow</td>
<td>5</td>
<td>3.40</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>3.50</td>
</tr>
<tr>
<td></td>
<td>7</td>
<td>3.65</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>3.75</td>
</tr>
<tr>
<td></td>
<td>9</td>
<td>3.85</td>
</tr>
<tr>
<td>Red</td>
<td>10 or more</td>
<td>4.00</td>
</tr>
</tbody>
</table>

As shown in Figure 2, the yellow zone is quite broad (five to nine exceptions). The penalty (raising the multiplier from 3 to 4) is automatically required for banks with ten or more exceptions. However, the penalty for banks with five to nine exceptions is subject to their supervisors’ discretions, based on what type of model error caused the exceptions. The Committee established four categories of causes for exceptions and guidance for supervisors for each category:

- **The basic integrity of the model is lacking.** Exceptions occurred because of incorrect data or errors in the model programming. The penalty should apply.
- **Model accuracy needs improvement.** The exceptions occurred because the model does not accurately describe risks. The penalty should apply.
- **Intraday trading activity.** The exceptions occurred due to trading activity (VaR is based on static portfolios). The penalty should be considered.
- **Bad luck.** The exceptions occurred because market conditions (volatility and correlations among financial instruments) significantly varied from an accepted norm. These exceptions should be expected to occur at least some of the time.

Although the yellow zone is broad, an accurate model could produce five exceptions 10.8% of the time. So even if a bank has an accurate model, it is subject to punishment over
10% of the time. Regulators are more concerned about the Type II errors since inaccurate models would have five errors 12.8% of the time (e.g., those with VaR calculated at the 97% confidence level rather than the required 99% confidence level). While this seems to be only a slight difference, using a 99% confidence level would result in a 1.24 times greater level of required capital, providing a powerful economic incentive for banks to use a lower confidence level.

Industry analysts have suggested lowering the required VaR confidence level to 95% and compensating by using a greater multiplier. This would result in a greater number of expected exceptions, and variances would be more statistically significant. The 1-year exception rate at the 95% level would be 13, and with more than 17 exceptions, the probability of a Type I error would be 12.5% (close to the 10.8 previously noted), but the probability of a Type II error at this level would fall to 7.4% (compared to 12.8% at a 97.5% confidence level). Thus, inaccurate models would be accepted less frequently.

Another way to make variations in the number of exceptions more significant would be to use a longer backtesting period. This approach may not be as practical, because the nature of markets, portfolios, and risk changes over time.

**Conditional Coverage**

AIM 5.5: Explain why it is necessary to consider conditional coverage in the backtesting framework.

We have been backtesting models based on unconditional coverage, in which the timing of our exceptions was not considered. In addition to having a predictable number of exceptions, we also anticipate the exceptions to be fairly equally distributed across time. A bunching of exceptions may indicate that market correlations have changed or that our trading positions have been altered.

We need some guide to determine if the bunching is random or caused by one of these changes. By including a measure of the independence of exceptions, we measure conditional coverage of the model. Christofferson² proposed extending the unconditional coverage test statistic (LRuc) to allow for potential time variation of the data. He developed a statistic to determine the serial independence of deviations using a log-likelihood ratio test (LRind). The overall log-likelihood test statistic for conditional coverage (LRcc) is then:

\[ LR_{cc} = LR_{uc} + LR_{ind} \]

We would reject the model if \( LR_{cc} > 5.99 \). If exceptions are determined to be serially dependent, then the model needs to be revised to incorporate the correlations that are evident in the current conditions.

1. Backtesting is an important part of VaR model validation. Backtesting involves comparing the number of instances when the actual profit/loss exceeds the VaR level (called exceptions) with the number predicted by the model at the chosen level of confidence.

2. Cleaned returns are generally used for backtesting. Cleaned returns are actual returns adjusted for all changes that arise from changes that are not mark to market.

3. The failure rate of a model backtest is the number of exceptions divided by the number of observations: \( N/T \).

4. The Basel Committee requires backtesting at the 99% confidence level over the past year (250 business days). At this level, we would expect \( 250 \times 0.01 \), or 2.5 exceptions.

5. In using backtesting to accept or reject a VaR model, we must balance the probabilities of two types of errors: a Type I error is rejecting an accurate model, and a Type II error is accepting an inaccurate model.

6. The Basel Committee rules establish zones of number of exceptions and corresponding penalties or increases in the capital requirement multiplier from 3 to 4 (i.e., safety factor).

7. Unconditional coverage testing does not evaluate the timing of exceptions, while conditional coverage tests review the number and timing of exceptions for independence. Current market or trading portfolio conditions may require changes in the model.
In backtesting a value at risk (VaR) model that was constructed using a 95% confidence level over a 255-day period, how many exceptions are forecasted?

A. 5.00.
B. 7.55.
C. 12.75.
D. 15.00.

Unconditional testing does not reflect the:

A. size of the portfolio.
B. number of exceptions.
C. confidence level chosen.
D. timing of the exceptions.

A Type I error occurs when:

A. accurate models are rejected.
B. accurate models are accepted.
C. inaccurate models are rejected.
D. inaccurate models are accepted.

Which of the following statements regarding verification of a VaR model by examining its failure rates is false?

A. The frequency of exceptions should correspond to the confidence level used for the model.
B. According to Kupiec (1995), we should reject the hypothesis that the model is correct if the LR > 3.84.
C. Backtesting VaR models with lower confidence levels is difficult because the number of exceptions is not high enough to provide meaningful information.
D. The range for the number of exceptions must strike a balance between the chances of rejecting an accurate model (a Type I error) and the chance of accepting an inaccurate model (a Type II error).

The Basel Committee has established four categories of causes for exceptions. Which of the following does not apply to one of those categories?

A. Small sample.
B. Intraday trading activity.
C. Model accuracy needs improvement.
D. The basic integrity of the model is lacking.
Cross Reference to GARP Assigned Reading – Jorion, Chapter 6

Concept Checker Answers

1. C \( (1 - 0.95) \times 255 = 12.75 \)

2. D Unconditional testing does not capture the timing of exceptions.

3. A A Type I error occurs when an accurate model is rejected.

4. C Backtesting VaR models with higher confidence levels is difficult because the number of exceptions is not high enough to provide meaningful information.

5. A Causes include the following: bad luck, intraday trading activity, model accuracy needs improvement, and the basic integrity of the model is lacking.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**VAR MAPPING**

**Exam Focus**

This topic introduces the concept of mapping a portfolio and shows how the risk of a complex, multi-asset portfolio can be separated into risk factors. We will examine the mapping of several types of portfolios, including fixed income portfolios and portfolios consisting of linear and nonlinear derivatives. Be able to describe the general mapping process and understand how it simplifies risk management for large portfolios.

**Mapping a Portfolio**

AIM 6.1: Explain the principles underlying VAR mapping, list and describe the mapping process.

Market risk is measured by first noting all of the current positions within a portfolio. These positions are then mapped to risk factors by means of factor exposures. Mapping involves finding common risk factors among positions in a given portfolio. If we have a portfolio consisting of a large number of positions, it may be difficult and time consuming to manage the risk of each individual position. Instead we can evaluate the value of these positions by mapping them onto common risk factors (e.g., changes in interest rates, changes in equity prices). By reducing the number of variables under consideration, we greatly simplify the risk management process.

Mapping can assist the risk manager in evaluating positions whose characteristics may change over time, such as with fixed income securities. Also, mapping can provide an effective way to manage risk when a sufficient history of data for an investment does not exist, such as with an initial public offering. In both cases, evaluating historical prices may not be relevant so the manager must evaluate those risk factors that are likely to impact the portfolio’s risk profile.

By cutting down the number of risk factors needed for analysis we can greatly reduce the amount of complexity. For example, when assessing risk factors we must also assess the correlation among risk factors which requires \( \frac{[n \times (n - 1)]}{2} \) covariance terms, where \( n \) is the number of risk factors. The number of parameters that are needed will grow exponentially with the number of risk factors, so mapping helps to simplify the process.

After finding the common risk factors, the risk manager constructs risk factor distributions and then inputs all data into the risk engine. This risk engine is responsible for deriving the profit/loss distribution of the portfolio returns. This distribution of portfolio returns can then be used to compute measures such as value at risk (VaR), which is the maximum loss over a defined period of time at a stated level of confidence.
When utilizing the risk management system, it is important to recognize that this system is **position-based** (a.k.a. holdings-based analysis). This position-based method differs from measuring risk with the more traditional return-based analysis where historical returns are evaluated over time. The return-based method is easy to implement, however, it suffers from the inability to incorporate new positions. As a result, **return-based analysis** may fail to spot style drift and/or hidden risks. The position-based method is necessary for alternative investments since some investments (e.g., emerging market funds) may have a short record of existence.

As mentioned, return-based analysis may fail to account for hidden risks. Risks are hidden when the calculation of performance measures such as the **Sharpe ratio** (i.e., excess return over volatility) indicates stable performance despite the possibility of a very large negative loss. For example, a portfolio that sells out-of-the-money put options on the S&P 500 index will generate stable returns when the options remain out-of-the-money. The returns are achieved from collecting the option premiums from the buyer. However, if the index falls in value and the options are exercised, this portfolio could suffer an infrequent, but large loss. This example illustrates how the returns-based approach can be misleading when assessing volatility. With a position-based approach, hidden risk is accounted for since all new positions and markets are incorporated into the risk measure.

Measuring and managing risk with a position-based analysis is clearly superior to return-based analysis, however, there are a few items that risk managers need to be aware of when implementing this method. Relative to a return-based system, it can be more expensive and require more resources since a portfolio could potentially be made up of hundreds of positions. In addition, this method assumes that the positions are frozen over the period of evaluation. This means that dynamic trading during the time period in question could produce misleading results. Finally, the model is only as good as the data supplied. As a result, **model risk** could be present if the portfolio-based system contains errors or approximations.

**Factor Exposures**

Factor exposures are an important component of any portfolio-based system and are widely used in risk measurement and management. They are necessary when mapping a portfolio to risk factors since dollar factor exposures are used to replace portfolio positions. When evaluating a fixed-income portfolio, for example, the change in interest rates is the appropriate risk factor. The appropriate exposure on this risk factor is **modified duration**. When combining duration and the change in interest rates, risk managers can easily obtain the percentage change in price for a given fixed-income position.

\[
\frac{\Delta P}{P} = -\text{modified duration} \times \Delta y
\]

Rearranging this formula allows for the computation of dollar duration, which is simply the position value multiplied by its modified duration.

\[
\Delta P = -(\text{modified duration} \times P) \times \Delta y
\]

\[
\Delta P = -(\text{dollar duration}) \times \Delta y
\]
This same process can also be applied to other risk factors and exposures. For example, the risk factor for equities would be the change in equity index prices. The corresponding factor exposure would be beta.

It is important to note that if all positions in a portfolio are exposed to the same risk factor, the portfolio factor exposure can be found as the weighted average of the position factor exposures. However, overall portfolio risk cannot be computed by aggregating exposures of different risk factors. For example, the modified duration of bonds cannot be combined with the beta of stocks when attempting to obtain the portfolio risk measure.

Another concern for risk manager is the potential for large movements in the risk factors. For example, with a large change in rates, duration will not be an appropriate measure of price change. In this case, second-order factor exposures such as convexity will be needed in order to more accurately reflect the actual change in bond price.

AIM 6.2: Explain how the mapping process captures general and specific risks.

So how many general (or primitive) risk factors are appropriate for a given portfolio? In some cases, one or two risk factors may be sufficient. Of course, the more risk factors chosen, the more time consuming the modeling of a portfolio becomes. However, more risk factors could lead to a better approximation of the portfolio’s risk exposure.

In our choice of general risk factors for use in VaR models, we should be aware that the types and number of risk factors we choose will have an effect on the size of residual or specific risks. Specific risks arise from the unsystematic risk of various positions in the portfolio. The more precisely we define risk, the smaller the specific risk will be. For example, a portfolio of bonds may include bonds of different ratings, terms, and currencies. If we use duration as our only risk factor, there will be a significant amount of variance among the bonds that we will call specific risk. If we add a risk factor for credit risk, we could expect that the amount of undefined specific risk would be smaller. If we add another risk factor for currencies, we would expect that the undefined specific risk would be even smaller.

Consider, for example, an equity portfolio with 5,000 stocks. Each stock has a market risk component and a firm-specific component. If each stock has a corresponding risk factor we would need roughly 12.5 million covariance terms (i.e., \([5,000 \times (5,000 - 1)] / 2\)) to evaluate the correlation between each risk factor. To simply the number of parameters required we need to understand that diversification will reduce the firm-specific components and leave only market risk (i.e., systematic risk or beta risk). We can then map the market risk component of each stock onto a stock index (i.e., changes in equity prices) to greatly reduce the number of parameters needed.
Mapping Approaches for Fixed-Income Portfolios

AIM 6.3: List and describe the three methods of mapping portfolios of fixed income securities.

After we have selected our general risk factors, we must map our portfolio onto these factors. There are three systems of mapping for fixed income securities:

1. **Principal mapping.** Includes only the risk of repayment of the principal amounts. This method considers the average maturity of the portfolio.

2. **Duration mapping.** The risk of the bond is mapped to a zero-coupon bond of the same duration. Duration mapping uses the duration of the portfolio to calculate the VaR.

3. **Cash flow mapping.** The risk of the bond is decomposed into the risk of each of the bonds' cash flows. Cash flow mapping is the most precise method because we map the present value of the cash flows (face amount discounted at the spot rate for that maturity) onto the risk factors for zeros of the same maturities and include the intermaturity correlations.

AIM 6.4: Map a fixed-income portfolio into positions of standard instruments.

The following two-position fixed-income portfolio will be used to illustrate the application of the fixed income systems:

- There is a short position in 1-year bonds with a $100 million face value and a 7% annual interest rate, with interest paid semiannually.
- There is a long position in 1-year bonds with a $1 billion face value and an 8% annual interest rate, with interest paid semiannually.
- The interest rate on zero-coupon bonds is 4.0% for 6-month maturity and 4.2% for 12-month maturity.

For principal mapping, VaR is calculated using the risk level from the zero-coupon bond equal to the average maturity of the portfolio. This method is the simplest of the three fixed income approaches.

For duration mapping, we calculate VaR by using the risk level of the zero-coupon bond equal to the duration of the portfolio. It may be difficult to calculate the risk level that exactly matches the duration of the portfolio.

For cash flow mapping, both the short and long positions will be decomposed into positions in two standard instruments consisting of a 6-month zero-coupon bond and a 12-month zero-coupon bond.

The short position will have the following negative cash flows:

- Cash flow 1: -$3.5 million interest in six months.
- Cash flow 2: -$3.5 million interest plus -$100 million principal, or -$103.5 million in 12 months.
The long position will have the following positive cash flows:

- Cash flow 1: $40 million interest in six months.
- Cash flow 2: $40 million interest plus $1 billion principal, or $1.04 billion in 12 months.

Both the short and long positions have now been decomposed into equivalent 6-month and 12-month zero-coupon bonds. They will now be mapped to interest rates on 6-month and 12-month zero-coupon bonds.

\[ x_i \text{ is defined as the standard instrument of the } i\text{th decomposed position, and there are four standard positions that have now been defined. } x_i \text{ equals the present value of the cash flow of its standard position.} \]

For the short position:

\[ x_1 = \frac{-3,500,000}{1 + (0.04/2)} = -3,431,373 \]

\[ x_2 = \frac{-103,500,000}{1 + 0.042} = -99,328,215 \]

For the long position:

\[ x_1 = \frac{40,000,000}{1 + (0.04/2)} = 39,215,686 \]

\[ x_2 = \frac{1,040,000,000}{1 + 0.042} = 998,080,614 \]

The portfolio has now been mapped to the 6- and 12-month zero interest rates. In order to calculate portfolio VaR, we would need to incorporate the correlations between the zeros. Cash flow mapping is the most precise method, but is also the most complex.

Professor's Note: The previous example illustrated how to map a portfolio onto standard instruments. The calculation of VaR after the portfolio has been cash flow mapped is a complicated process that is unlikely to show up on the exam.

AIM 6.5: Discuss how mapping of risk factors can support stress testing.

If we assume that there is perfect correlation among maturities of the zeros, the portfolio VaR would be equal to the undiversified VaR (i.e., just the sum of the VaRs). Instead of calculating the undiversified VaR directly, we could reduce each zero-coupon value by its respective VaR and then revalue the portfolio. The difference between the revalued portfolio and the original portfolio value should be equal to the undiversified VaR. Stressing each zero by its VaR is a simpler approach than incorporating correlations, however, this method ceases to be viable if correlations are anything but perfect (i.e., 1).
Benchmarking a Portfolio

AIM 6.6: Explain how VaR can be used as a performance benchmark.

It is often convenient to measure VaR relative to that of a benchmark portfolio. This is what is referred to as benchmarking a portfolio. Portfolios can be constructed that match the risk factors of the benchmark portfolio but have either a higher or a lower VaR. The VaR of the deviation between the two portfolios is referred to as a tracking error VaR. In other words, tracking error VaR is a measure of the difference between the VaR of the target portfolio and the benchmark portfolio.

Mapping Approaches for Linear Derivatives

AIM 6.7: Describe the method of mapping forwards, commodity forwards, forward rate agreements, and interest-rate swaps.

The delta-normal method provides accurate estimates of VaR for portfolios and assets that can be expressed as linear combinations of normally distributed risk factors. Once a portfolio, or financial instrument, is expressed as a linear combination of risk factors, a covariance (correlation) matrix may be generated, and VaR can be measured using matrix multiplication.

Forwards are appropriate for application of the delta-normal method. Their values are a linear combination of a few general risk factors, which have commonly available volatility and correlation data.

To illustrate this idea, consider a forward contract to purchase pounds for dollars one year from now. This forward position is analogous to three separate risk positions:

3. A long position in the British pound spot market.

Given the volatilities of each of these positions and the correlation matrix for these positions along with the cash flows, the VaR for the individual risk positions, and the component VaR for each position, the VaR of the forward contract can be determined using matrix algebra.

The general procedure we've outlined applies to other types of financial instruments, such as forward rate agreements and interest rate swaps. As long as an instrument can be expressed as linear combinations of its basic components, the delta-normal VaR may be applied with reasonable accuracy.
AIM 6.8: Describe the method of mapping options.

As you are aware by now, the delta-normal VaR method is based on linear relationships between variables. Options, however, exhibit nonlinear relationships between movements of the values of the underlying instruments and the values of the options. In many cases, the delta-normal method may still be applied because the value of an option may be expressed linearly as the product of the option delta and the underlying asset.

Unfortunately, the delta-normal VaR cannot be expected to provide an accurate estimate of true VaR over ranges where deltas are unstable. Deep out-of-the-money and deep in-the-money options have relatively stable deltas. Over these ranges, the relationship between the value of the underlying instrument and the value of the option is very much like a forward currency contract. The delta-normal VaR can be calculated by assessing the volatility of the underlying spot prices and the correlation between the price of the option and the spot price.

Professor's Note: Options are usually mapped by using Taylor series approximation and using the delta-gamma method to calculate the option VaR.
Cross Reference to GARP Assigned Reading – Jorion, Chapter 11

**Key Concepts**

1. Portfolio exposures are broken down into general risk factors and mapped onto those factors.
2. Undefined specific risk decreases as more risk factors are added to a VaR model.
3. Fixed income risk mapping methods include principal mapping, duration mapping, and cash flow mapping. Each provides a different level of complexity and precision.
4. Short and long positions in the portfolio can be decomposed and mapped to standard instruments, such as 6-month and 12-month zero-coupon bonds.
5. A popular use of VaR is to establish a benchmark portfolio and measure VaR of other portfolios in relation to this benchmark.
6. Delta-normal VaR can be applied to portfolios of many types of instruments as long as the risk factors are linearly related.
7. Application of the delta-normal method with options and other derivatives does not provide accurate VaR measures over ranges in which deltas are unstable.
1. Delta-normal VaR will provide accurate estimates for option contracts when:
   A. deltas are stable.
   B. options are at-the-money.
   C. the correlation matrix is available.
   D. the delta-normal method can never be used for option contracts.

2. There is a short position in 1-year bonds with a $150 million face value and a 6% annual interest rate, with interest paid semiannually. The annualized interest rate on zero-coupon bonds is 3.8% for a 6-month maturity and 4.1% for a 12-month maturity. Decompose the bond into the cash flows of the two standard instruments, and then determine the present value of the cash flows of the standard instruments. What are the present values of each cash flow?

<table>
<thead>
<tr>
<th>PV of CF1</th>
<th>PV of CF2</th>
</tr>
</thead>
<tbody>
<tr>
<td>-$4,117,945</td>
<td>-$139,882,651</td>
</tr>
<tr>
<td>-$4,226,094</td>
<td>-$143,873,919</td>
</tr>
<tr>
<td>-$4,416,094</td>
<td>-$148,414,986</td>
</tr>
<tr>
<td>-$4,879,542</td>
<td>-$144,224,783</td>
</tr>
</tbody>
</table>

3. Which of the following is not one of the three approaches for mapping portfolio fixed income securities onto risk factors?
   A. Principal mapping.
   B. Duration mapping.
   C. Cash flow mapping.
   D. Present value mapping.

4. If portfolio assets are perfectly correlated, portfolio VaR will equal:
   A. marginal VaR.
   B. component VaR.
   C. undiversified VaR.
   D. diversified VaR.

5. Which of the following can be considered a general risk factor?
   I. Exchange rates.
   II. Zero-coupon bond.
   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.
1. A Delta-normal VaR methods will provide accurate estimates of VaR for options only over those ranges in which the deltas of the contracts are stable. Deltas are normally unstable near the money and close to expiration.

2. C The standard instruments are $-150,000,000 \times (0.06 / 2) = -4,500,000$ for six months, and $-4,500,000 - 150,000,000 = -154,500,000$ for 12 months. The present values are $-4,500,000 / 1.019 = -4,416,094$, and $-154,500,000 / 1.041 = -148,414,986$.

3. D Present value mapping is not one of the approaches. Each of the others is used.

4. C If we assume perfect correlation among assets, VaR would be equal to undiversified VaR.

5. A Exchange rates can be used as general risk factors. Zero-coupon bonds are used to map bond positions, but are not considered a risk factor. However, the interest rate on those zeroes is a risk factor.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP. This topic is also covered in:

**MEASURES OF PRICE SENSITIVITY BASED ON PARALLEL YIELD SHIFTS**

*Exam Focus*

Duration and convexity measure bond price volatility for a one-time parallel shift in all yield curves. This limits the use of these measures. Moreover, yield is not a good measure of potential return since it assumes all coupons are reinvested at that rate. These realities become evident when we analyze nonparallel changes in the yield curve. This topic presents alternative measures of price volatility that capture the sensitivity of bond prices to parallel shifts in the yield curve. Yield-based DV01, a special form of DV01, assumes that the yield is the interest rate factor and the pricing function is the price-yield relationship. Modified and Macaulay duration are presented as alternatives to effective duration.

AIM 7.1: Describe advantages, disadvantages, and limitations of the use of price sensitivities based on parallel shifts of the yield curve.

Measures of price sensitivity which are based on parallel yield curve shifts are widely used in the financial sector. Measures such as modified and Macaulay duration are easy to compute and understand. In addition, a general understanding of measures based on parallel yield curve shifts often make other price sensitivity measures more intuitive. One limitation is the impractical assumption that all yields will shift in a parallel fashion. Another limitation is that these measures assume cash flows are fixed. In this topic, we will examine the calculation of three price sensitivity measures: yield-based DV01, modified duration, and Macaulay duration.

AIM 7.2: Define and calculate yield-based DV01, modified duration, and Macaulay duration.

**YIELD-BASED DV01**

The *yield-based DV01* is similar to the DV01 measure from Part I of the FRM Program, but it explicitly assumes that the yield of a security is the interest rate factor and that the pricing function is the price-yield relationship.

Yield-based DV01 is calculated as follows:

\[
yield-based DV01 = \left( \frac{1}{10,000} \right) \times \left( \frac{1}{1 + \text{periodic yield}} \right) \times \text{sum of time-weighted present values of the bond's cash flows}
\]
The yield-based DV01 can be interpreted as the dollar change in bond price from a 1 basis point change in yield (remember that price and yield are inversely related). For example, suppose a bond has DV01 = 0.08. A 1 basis point increase in yield will cause a $0.08 per $100 par value decrease in the bond price.

**Modified and Macaulay Duration**

**Modified duration** is another special case of duration that can be interpreted as the approximate percentage change in the price of a bond from any given change in yield, which is also the elasticity of the bond price with respect to yield. Modified duration is similar to yield-based DV01, except bond price replaces 10,000 in the formula:

\[
\text{modified duration} = \left( \frac{1}{P} \right) \times \left( \frac{1}{1 + \text{periodic yield}} \right) \times \left( \text{sum of the time-weighted present values of the bond's cash flows} \right)
\]

where:

\[ P = \text{bond price} \]

**Macaulay duration** is the weighted average term to maturity of a bond's cash flows. It is measured in the number of periods (usually years). To compute Macaulay duration, use the following relationship:

\[
\text{Macaulay duration} = \left( 1 + \frac{y}{2} \right) \times \text{modified duration}
\]

*Professor's Note: Recall that for a semiannual coupon paying bond, the periodic yield would be: annual yield, \( y \), divided by 2.*

**Example: Computing yield-based DV01, modified duration, and Macaulay duration**

Compute the yield-based DV01, the modified duration, and the Macaulay duration of a 3-year, 4%, semiannual coupon bond yielding 3.5% on a bond-equivalent basis.

**Answer:**

**Figure 1: Bond Data**

<table>
<thead>
<tr>
<th>Term (( j ))</th>
<th>Cash Flow ($)</th>
<th>( PVCF_j )</th>
<th>( j \times PVCF_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>2</td>
<td>1.9656</td>
<td>0.98</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.9318</td>
<td>1.93</td>
</tr>
<tr>
<td>1.5</td>
<td>2</td>
<td>1.8986</td>
<td>2.85</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.8659</td>
<td>3.73</td>
</tr>
<tr>
<td>2.5</td>
<td>2</td>
<td>1.8338</td>
<td>4.58</td>
</tr>
<tr>
<td>3</td>
<td>102</td>
<td>91.9165</td>
<td>275.75</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>112</strong></td>
<td><strong>101,4122</strong></td>
<td><strong>289.82</strong></td>
</tr>
</tbody>
</table>
Similar to the dollar value of a basis point (DV01), duration is a linear approximation to the convex price/yield relationship. As “Δy” increases, the accuracy of the PPC approximation will decrease.

Duration measures have characteristics similar to the price value of a basis point and other price volatility measures. All else equal, as:

- Yield decreases (price increases), duration increases.
- Yield increases (price decreases), duration decreases.
- Coupon declines, duration increases.
- Maturity increases, duration increases.

**Effective Duration**

In comparing the various duration measures, both Macaulay and modified duration are calculated directly from the promised cash flows for a bond with no adjustment for the effect of any embedded options on cash flows. Effective duration is calculated from expected price changes in response to changes in yield that explicitly take into account a bond’s option provisions. For option-free bonds, however, effective duration (based on small changes in yield) and modified duration will be very similar.
MACAULAY DURATION, COUPON RATES, AND MATURITY

AIM 7.3: Calculate and describe the Macaulay duration of zero-coupon bonds, par bonds, and perpetuities.

Figure 2 illustrates the relationship between Macaulay duration, coupon rate, and maturity for bonds priced to yield 5%. As indicated, Macaulay duration decreases as coupon rate increases. Also apparent in Figure 2 is that the Macaulay duration for a zero-coupon bond is its maturity, and the duration of a par bond approaches that of a perpetuity as maturity increases. This relationship makes sense because higher coupon bonds have a greater percentage of their value paid out earlier, as opposed to zero-coupon bonds, where the whole value is a single payment at maturity.

The thing to remember here is that long-term, zero-coupon bonds have the greatest duration, regardless of whether we are talking about Macaulay or modified duration.

Figure 2: Macaulay Duration, Coupon Rate, and Maturity

DV01, COUPON RATES, AND MATURITY

AIM 7.4: Explain how coupon rate, maturity, and yield impact the duration and DV01 of a fixed income security.

Figure 3 illustrates the relationship between DV01, coupon rate, and maturity for bonds priced to yield 5%. From Figure 3, it is easy to see that the DV01 of a bond increases with its coupon rate and that the DV01 of par and premium bonds continuously increases with maturity. Additionally, it is apparent that the rate of increase in DV01 for the premium bond, relative to maturity, is greater than the rate at which the DV01 of the par bond increases. This is because the price of the premium bond increases with maturity, and price and DV01 are positively related. Also shown in Figure 3 is that the DV01 of a zero-coupon
bond approaches zero as maturity increases because the price of the bond approaches zero as maturity increases—if price equals zero, so must DV01.

**Figure 3: DV01, Coupon Rate, and Maturity**

![Figure 3: DV01, Coupon Rate, and Maturity](image)

**DV01, Duration, and Yield**

AIM 7.5: Define DV01 in terms of Macaulay duration and use this definition to explain and differentiate between the “duration” effect and the “price” effect.

The dollar value of a basis point can be expressed in terms of modified and Macaulay duration as follows.

\[
\text{DV01} = \frac{\text{bond value} \times \text{duration}_{\text{modified}}}{10,000}
\]

\[
\text{DV01} = \frac{\text{bond value} \times \text{duration}_{\text{Macaulay}}}{10,000 \times (1 + \text{periodic yield})}
\]

As demonstrated from the above equations as well as Figure 3, increasing duration will almost always increase the DV01 measure and hence the price sensitivity of a bond. This increasing sensitivity from duration is known as the **duration effect**. At some point along the maturity spectrum, the price of a bond will play a bigger role and force the DV01 to drop such as the deep discount and zero coupon bonds in Figure 3. The changing sensitivity due to bond value is known as the **price effect**. The price effect can also be demonstrated by viewing the path of the premium bond. In this case, the duration effect has combined with the price effect to increase the price sensitivity above that of a par bond.

From the equations, we can see that increasing period yield will lower the DV01. Increasing yields will also lower the duration. As the yield increases, the present value of the cash flows is lowered, with the cash flows further out being affected the most. The relationship is fairly obvious by inspecting the formula for the DV01, as yield is always in the denominator.
YIELD-BASED CONVEXITY AND MATURITY

AIM 7.6: Define yield-based convexity and explain how yield-based convexity changes for changes in maturity.

Yield-based convexity is sometimes referred to as just simply convexity as it is the second derivative of the price-yield function. Convexity can be calculated using an approach that time-weights the present value of the bond’s cash flows (much like yield-based DVOI). However, a simpler method is to use the formula for convexity outlined in the FRM Part I readings:

\[ \text{convexity} = \frac{\text{BV}_{-\Delta y} + \text{BV}_{+\Delta y} - 2 \times \text{BV}_0}{\text{BV}_0 \times \Delta y^2} \]

Both methods will produce similar results.

To gain a better understanding of the effects of maturity on convexity, we can evaluate the changes in the convexity of a zero-coupon bond over time. The convexity of a zero coupon bond is calculated as:

\[ \text{convexity}_{\text{zero}} = \frac{T(T + 0.5)}{\left(1 + \frac{y}{2}\right)^2} \]

From this equation it is easy to see that longer maturity zeros will have more of an impact on convexity than shorter maturity zeros. In fact, we can say that, holding all else constant, convexity will increase with the square of maturity. Furthermore, since a coupon bond is essentially a package of zero-coupon bonds, the convexity of longer-term coupon bonds is generally greater than the convexity of shorter-term coupon bonds.

CONVEXITY, BARBELLS, AND BULLETS

AIM 7.7: Explain the difference between a barbell and a bullet portfolio and analyze the impact convexity may have on both.

In portfolio management, a barbell strategy is when a manager uses bonds with short and long maturities—forgoing any intermediate-term bonds. A bullet strategy is when portfolio managers buy bonds concentrated in the intermediate maturity range.

Since duration is linearly related to maturity, it is possible for a bullet and a barbell strategy to have the same duration. However, since convexity increases with the square of maturity, these two portfolios will have different convexities. For example, consider two portfolios: one a bullet strategy, the other a barbell strategy with maturities above and below the average maturity (duration) of the bullet strategy. In this case, the barbell will have the greater convexity due to the exponential (squared) influence of the longer-term bonds.
Key Concepts

1. Macaulay duration has the following characteristics:
   • When measured in periods, it is the weighted average term to maturity of the bond’s cash flows.
   • It is related to price volatility since it can be interpreted as the percentage price change in the bond (PPC) from a 1 percentage-point change in yield.
   • It is equal to 1 plus the periodic yield times the modified duration (MD).

2. The yield-based DV01 is similar to the DV01. It assumes the yield of a security is the interest rate factor and the pricing function is the price-yield relationship.

   \[
   \text{yield-based DV01} = \left( \frac{1}{10,000} \right) \times \left( \frac{1}{1 + \text{periodic yield}} \right) \times \left( \frac{\text{sum of time-weighted present values of the bond’s cash flows}}{100} \right)
   \]

3. Convexity increases with the square of maturity, holding all else constant.

4. The convexity of a barbell strategy is higher than that of a bullet strategy with equal duration because duration increases linearly with maturity, while convexity increases with the square of maturity.
1. The formula for yield-based DV01 can be represented as:
   A. modified duration times the quantity 1 plus the semiannual yield.
   B. sum of time-weighted present value of bond cash flows times 1/10,000 times the reciprocal of the quantity 1 plus the semiannual yield.
   C. time-weighted present value of bond cash flows times reciprocal of bond price times the reciprocal of the quantity 1 plus the semiannual yield.
   D. modified duration divided by the quantity 1 plus the annual yield.

2. A bond has a modified duration of 2.2 years. If the yield decreases by 0.5%, the percentage change in the price of the bond will be a:
   A. 2.2% increase.
   B. 2.2% decrease.
   C. 1.1% increase.
   D. 1.1% decrease.

3. A bond with a 10-year maturity and a 5% semiannual coupon has a Macaulay duration of 8.0356. The current yield on the bond is 4.5%. What is the modified duration of the bond?
   A. 7.6896.
   B. 7.8588.
   C. 8.0356.
   D. 8.2164.

4. A bond portfolio consists of five bonds:
   - Bond 1: 5%, annual-pay bond with a 10-year maturity and a yield of 4.5%.
   - Bond 2: 5%, semiannual-pay bond with a 10-year maturity and a yield of 4.5%.
   - Bond 3: A zero-coupon bond with a 10-year maturity and a yield of 4.5%.
   - Bond 4: 4%, semiannual-pay bond with a 10-year maturity and a yield of 4.5%.
   - Bond 5: 5%, annual-pay bond with a 10-year maturity and a yield of 5.5%.

Which of the following statements about these bonds is correct?
   A. Bond 1 has a shorter duration than Bond 2.
   B. The Macaulay duration of Bond 3 is five years.
   C. Bond 4 has a shorter duration than Bond 2.
   D. The DV01 of Bond 5 is lower than the DV01 of Bond 1.
5. Given the following bond portfolios:

<table>
<thead>
<tr>
<th>Bond Maturity</th>
<th>Portfolio 1 Duration Contribution</th>
<th>Portfolio 2 Duration Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-year bonds</td>
<td>1.32</td>
<td>0.52</td>
</tr>
<tr>
<td>5-year bonds</td>
<td>1.37</td>
<td>3.18</td>
</tr>
<tr>
<td>10-year bonds</td>
<td>3.95</td>
<td>1.05</td>
</tr>
<tr>
<td>20-year bonds</td>
<td>1.51</td>
<td>3.40</td>
</tr>
<tr>
<td>Effective portfolio duration</td>
<td>?</td>
<td>?</td>
</tr>
</tbody>
</table>

Which of the following statements is correct?
A. Portfolio 1 is a barbell portfolio.
B. Portfolio 2 is a bullet portfolio.
C. It is impossible for Portfolios 1 and 2 to have the same duration.
D. Portfolio 2 will have greater convexity than Portfolio 1.
1. B yield-based DV01 = \( \frac{1}{10,000} \times \frac{1}{1 + \frac{y}{2}} \times \text{sum of the time-weighted present values of the bond’s cash flows} \)
   where:
   \( y \) = annual yield
2. C approximate PPC = \(-2.2 \times -0.5\% = 1.1\% \) increase
3. B Macaulay duration = \( 1 + \frac{y}{2} \times \text{modified duration} \)
   \[ 8.0356 = \left( 1 + \frac{0.045}{2} \right) \times \text{modified duration} \]
   modified duration = \( \frac{8.0356}{1.0225} = 7.8588 \)
4. D Choice D is correct. Increasing the yield will lower the DV01. Since Bond 5 has a higher yield than Bond 1, it must have a lower DV01. Choice B is incorrect. The Macaulay duration of a zero-coupon bond will be equal to its maturity. Choices A and C are incorrect. All else equal, a semiannual-pay bond will have a shorter duration than an annual-pay bond, so Bond 2 has a shorter duration than Bond 1. A premium bond will have a shorter duration than a discount bond, so Bond 2 will have a shorter duration than Bond 4.
5. D Since Portfolio 2 has more long-term bonds than short-term bonds and since convexity is related to the square of maturity, Portfolio 2 will have greater convexity. The other statements are incorrect. Portfolio 1 is a bullet portfolio (concentrated in intermediate maturities), and Portfolio 2 is a barbell. It is possible for a bullet and a barbell to have the same duration. In fact, adding the duration contribution of both portfolios gives a duration value of 8.15.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**Key Rate and Bucket Exposures**

**Exam Focus**

This topic introduces the idea that multiple factors can be analyzed when hedging fixed income portfolios. The two techniques discussed are key rate shifts and bucket shifts. Both methods allow the user to hedge a position against changes in rates in certain areas of the maturity spectrum. The key rate method is straightforward and assumes that rates change in the region of the key rate chosen. The bucket method is similar but uses information from a greater array of rates, specifically those built into the forward rate curve. Investors can generate a better hedge using these techniques, but since both are based on duration, the hedge will not perform as anticipated if rates ultimately do not change as expected.

**Single-Factor Approaches**

AIM 8.1: Describe and analyze the major weakness attributable to single-factor approaches when hedging portfolios or implementing asset liability techniques.

The major weakness of a single-factor model approach to hedging or asset/liability management is the assumption that the entire yield curve can be described by one interest rate factor. For instance, this assumption indicates that changes in the 2-year interest rate instrument can be used to describe changes in 6-month instruments, as well as 15-year or 20-year instruments. If a manager is using the yield-based duration of an intermediate-term asset to protect against changes in both short and long liabilities, and yields do not change in a parallel fashion, the intermediate instrument will fail to change in the appropriate manner to match changing values in the liability structure. In sum, a single-factor-based approach to hedging or asset/liability management will not protect against changes in nonparallel shifts in the shape of the yield curve. A multiple-factor approach is needed when changes in the shape of the yield curve occur.

**Key Rate Shift Technique**

AIM 8.2: Describe key-rate shift analysis.

The key rate shift technique is an approach to nonparallel shifts in the yield curve that allows for changes in all rates to be determined by changes given in key rates. In order to apply the key rate technique, choices have to be made with respect to the number of key rates chosen, the type of rate to be chosen (spot or par yield), the term of each key rate, and the rule for computing other rates given changes in the key rates. An example may be helpful in describing this technique.
Assume there are three key rates: the 1-year, 7-year, and 20-year spot yields. The key rate technique indicates that changes in each key rate will affect rates from the term of the previous key rate to the term of the subsequent key rate. In this case, the 1-year key rate will affect all rates from 0 to 7 years; the 7-year affects all rates from 1 year to 20 years; and the 20-year affects all rates from 7 years to the end of the curve. If one assumes a simplistic one basis point effect, the impact of each key rate will be one basis point at each key rate and then a linear decline to the subsequent key rate. This key rate shift behavior is illustrated in Figure 1.

This type of approach has four appealing characteristics: (1) key rates are affected by a combination of rates closest to it; (2) key rates are mostly affected by the key rate which is closest to it; (3) key rate effects are smooth, meaning impacts do not jump across maturity; and (4) a parallel shift across the yield curve results. Although a linear relationship incorporates changes in rates across key rates, any relationship could have been used (e.g., curves). However, incorporating complexity in the rate relationships has not been shown to add much value.

So how many key rates are appropriate when implementing a hedging strategy based on key rates? Obviously, a higher number of key rates will produce a higher quality hedge. However, additional key rates will also require higher trading costs as well as more time in order to properly maintain the hedge. As a result, it is ideal to spread out the key rates to match the desired maturities. For instance, it would be more convenient to hedge with key rate securities that are very liquid.
KEY RATE EXPOSURE

AIM 8.3: Define, calculate, and interpret key rate 01 and key rate duration.

Suppose a 30-year semiannual-paying noncallable bond pays a $4,500 semiannual coupon in a flat spot rate environment of 5% across all maturities. If we assume a one basis point shift in the spot key rates used (2, 5, 10 and 30 year key rates), the subsequent key rate effects on the security are as shown in Figure 2.

Figure 2: Key Rate Exposure of 30-Year Semiannual-Pay Noncallable Bond

<table>
<thead>
<tr>
<th></th>
<th>Value</th>
<th>Key Rate 01 ($)</th>
<th>Key Rate Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial value</td>
<td>$139,088.95</td>
<td></td>
<td></td>
</tr>
<tr>
<td>After 2-yr shift</td>
<td>139,083.96</td>
<td>4.99</td>
<td>0.36</td>
</tr>
<tr>
<td>After 5-yr shift</td>
<td>139,074.21</td>
<td>14.74</td>
<td>1.06</td>
</tr>
<tr>
<td>After 10-yr shift</td>
<td>139,015.04</td>
<td>73.91</td>
<td>5.31</td>
</tr>
<tr>
<td>After 30-yr shift</td>
<td>139,024.26</td>
<td>64.70</td>
<td>4.65</td>
</tr>
<tr>
<td>Total</td>
<td>158.35</td>
<td></td>
<td>11.38</td>
</tr>
</tbody>
</table>

The value column of Figure 2 shows the exposure effect of changing the key rates, 2-year, 5-year, 10-year, and 30-year on the value of the security. The key rate 01 ($) column shows the effect of a dollar change of a one-basis-point shift around each key rate on the value of the security. Key rate duration can be interpreted as the approximate percentage change in the value of a bond or bond portfolio in response to a 100-basis-point change in a given key rate, holding all other rates constant. This column is calculated by dividing key rate 01 by the initial value times 10,000. For example:

\[
\text{key rate duration}_{5\text{-year shift}} = \frac{14.74}{139,088.95} \times 10,000 = 1.06
\]

The sum of the key rate durations will equal the effective duration of the 30-year bond.

The only way to calculate the key rate 01 exposures is to adjust each of the rates as illustrated by the diagram in Figure 1 (note that different key rates are used for this example). This is a tedious process, but the important piece of information comes from the sensitivity of exposure to shifts in each key rate. These key rate exposures would allow us to better hedge our position to exposures in each key rate than by simply assuming a parallel shift across all maturities in the maturity spectrum.
HEDGING BASED ON KEY RATES

AIM 8.4: Describe the key rate exposure technique in multifactor hedging applications and discuss its advantages and disadvantages.

AIM 8.5: Calculate the key rate exposures for a given security, and compute the appropriate hedging positions given a specific key rate exposure profile.

To illustrate hedging based on key rates, suppose that four other securities exist in addition to the noncallable bond just discussed and that each of these new securities have the following key rate exposures:

- The 2-year security only has a 2-year key rate exposure of 0.015 per $100 face value.
- The 5-year security has exposures over the 2-year and 5-year key rate of 0.0025 and 0.035, respectively, per $100 face value.
- The 10-year security has exposures over the 2-year, 5-year, and 10-year key rates of 0.003, 0.015, and 0.1, respectively, per $100 face value.
- The 30-year security only has exposure to the 30-year key rate of 0.15 per $100 face value.

It is assumed in this example that the 2-year bond and the 30-year bond are trading at par, so their only exposure is to the key rate corresponding to the maturity date. Using the previous example's key rate exposures generates the following set of equations to establish the hedge:

\[
\begin{align*}
2\text{-year key rate exposure:} & \quad \frac{0.015}{100} \times F_2 + \frac{0.0025}{100} F_5 + \frac{0.003}{100} F_{10} = 4.99 \\
5\text{-year key rate exposure:} & \quad \frac{0.035}{100} \times F_3 + \frac{0.015}{100} F_{10} = 14.74 \\
10\text{-year key rate exposure:} & \quad \frac{0.1}{100} F_{10} = 73.91 \\
30\text{-year key rate exposure:} & \quad \frac{0.15}{100} \times F_{30} = 64.70
\end{align*}
\]

By simultaneously solving for \( F_2 \), \( F_3 \), \( F_{10} \), and \( F_{30} \), these equations indicate that one needs to short the 2-year security in the amount of $16,745 worth of face, short the 5-year in the amount of $10,439 worth of face, short the 10-year in the amount of $73,910 worth of face, and short the 30-year in the amount of $43,133 worth of face value. Combining these short positions with the 30-year position from the previous example will immunize the portfolio from changes in rates close to the key rates selected.
AIM 8.7: Discuss why hedges based on key rates only approximate an immunized position in the underlying assets.

As is the case with most duration-based hedging techniques, the assumption of interest rate movements drives the effectiveness of immunized strategies. There are two factors at work when using key rates in an immunization-type setting. If interest rates change more dramatically than indicated, the immunized position will not perform as expected. This nonperformance will be exacerbated given larger changes in interest rates. More importantly, however, is the assumption of how interest rates will change around and between key rates. If the assumed rate shifts do not change in accordance with the assumed path indicated by the key rate technique, the effectiveness of the immunized position will be decreased. Losses or gains will accrue, which will directly affect the immunization strategy.

Simply stated, using the key rates in an immunized setting will only be an approximation of the effectiveness of immunization. This is a direct result of the dependence of the technique upon the ultimate size and movement of rates in and around the key rates chosen. The only way immunization will work perfectly in a real-world setting is if all sources of interest rate changes are perfectly matched. This is obviously a difficult task to complete.

**Key Rate Shift vs. Bucket Shift Approaches**

AIM 8.8: Describe the relationship between key rate and bucket exposures.

AIM 8.9: Explain the main differences between the key rate shift and the bucket shift approach to manage interest rate risks.

The two main differences between the key rate shift and bucket shift approaches relate to the number of interest rate factors used in the analysis and the assumption of the movement in these factors. Key rate shifts incorporate a relatively small number of key rates in its analysis, whereas the bucket shift approach uses many potential effects within a region of the yield curve. For instance, the key rate shift approach may use the 2-year and 5-year to approximate interest rate effects. The bucket shift approach, on the other hand, would incorporate specific assumptions about rate shifts with payment periods up to the 5-year horizon. The attention paid to each payment period would more effectively approximate changes, but would also require a more detailed analysis of the potential effects along the chosen region.

The second major difference between the two approaches relates to the assumption of movement in interest rates. The key rate shift approach assumes changes in rates in and around the chosen key rates, while the bucket shift approach assumes parallel changes in the forward rates implicit in the region of the curve being investigated. This makes the bucket shift approach more appropriate for managing the interest rate risk of a swaps portfolio.
AIM 8.10: Explain how key rate and bucket analysis may be applied in estimating portfolio volatility.

Recall that key rates and bucket analysis allow a manager to use more than a single factor to manage interest rate risk effects on a portfolio. This multifactor approach works well not only in estimating changes in the level of the portfolio, but also in the estimation of portfolio volatility because it allows for the incorporation of correlation effects between various interest rate assumptions.

Suppose one has information related to the volatility effects of two key rates. In this case, a manager can use traditional portfolio volatility relationships not only to incorporate the volatility impacts of each individual key rate, but also to incorporate the correlation between each key rate. The bucket technique works in a similar fashion, but because it is based on estimating forward rate effects, the number of inputs and correlation pairs that must be incorporated is greater.
1. Using single-factor models when attempting to hedge portfolios generally falls short of the hedged expectations because they assume that one interest rate can indicate changes in interest rates across the maturity spectrum. Multifactor approaches are preferred over single-factor approaches when hedging.

2. The key rate shift technique is a multifactor approach to nonparallel shifts in the yield curve that allows for changes in all rates to be determined by changes given in key rates. Choices have to be made regarding which key rates shift and how the key rate movements relate to prior or subsequent maturity key rates.

3. Calculating key rate exposures requires the analyst to generate new prices of securities, assuming changes in each key rate individually. The overall exposure to shifts in key rates results in the portfolio exposure over shifts across all key rates.

4. Hedging positions can be created in response to shifts in key rates by equating individual key rate exposures adjacent to key rate shifts to the overall key rate exposure for that particular key rate change. The resulting positions indicate either long or short positions in securities to protect against interest rate changes surrounding key rate shifts.

5. Using the key rate shift technique requires a reliance on duration-based concepts. Hence, any and all hedged positions will be sensitive to differential changes in portfolio value due to shifting duration exposures. As with all hedging situations, portfolios must be monitored for shifts in duration sensitivities, especially in the presence of large interest rate changes.

6. The key rate and bucket rate methods differ from the number of inputs used in their analyses. Key rates focus on a specified number of rates, while bucket shifting focuses on effects specific to a certain region on the maturity spectrum.

7. Multifactor approaches to hedging, such as key rate and bucket shift approaches, can also be used to estimate portfolio volatility effects because they incorporate correlations across a variety of interest rate effects.
### Concept Checkers

1. The main problem associated with using single-factor approaches to hedge interest rate risk is:
   A. no method can hedge interest rate risk.
   B. single-factor models assume mean-reversion between one short-term and one long-term rate.
   C. single-factor models assume effects across the entire curve dictated by one rate.
   D. single-factor models assume risk-free securities have credit exposure.

2. Using key rates of 2-year, 5-year, 7-year, and 20-year exposures assures all of the following except that the:
   A. 2-year rate will affect the 5-year rate.
   B. 7-year rate will affect the 20-year rate.
   C. 5-year rate will affect the 7-year rate.
   D. 2-year rate will affect the 20-year rate.

3. Using either key rates or bucket exposures in hedging is likely to perform as expected when rates:
   A. do not change as expected.
   B. change in a larger amount than expected.
   C. change as expected.
   D. change inversely as forecasted.

4. Assume you own a security with a 2-year key rate exposure of $4.78, and you would like to hedge your position with a security that has a corresponding 2-year key rate exposure of 0.67 per $100 of face value. What amount of face value would be used to hedge the 2-year exposure?
   A. $478.
   B. $239.
   C. $713.
   D. $670.

5. Which of the following differences between key rate and bucket analysis is true?
   I. Estimating portfolio volatility with both methods is similar except the bucket technique requires less inputs and correlations.
   II. The key rate shift approach assumes changes in rates in and around the chosen key rates.
   A. I only.
   B. II only.
   C. Both I and II.
   D. Neither I nor II.
CONCEPT CHECKER ANSWERS

1. C  Single-factor models assume that any change in any rate across the maturity spectrum can indicate changes across any other portion of the curve.

2. D  Key rate exposures assume that key rates chosen adjacent to the rate of interest are affected, not across other key rates.

3. C  Using key rates and bucket exposures assume rates shift as expected. If they do not, any hedge that uses either technique will result in some interest rate risk remaining.

4. C \[
\frac{0.67}{100} F = $4.78 \\
F = $713.43
\]

5. B  Estimating portfolio volatility with both methods is similar except the bucket technique requires more inputs and correlations. The key rate shift approach assumes changes in rates in and around the chosen key rates.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

THE SCIENCE OF TERM STRUCTURE MODELS

EXAM FOCUS

The emphasis of this topic is the pricing of interest rate derivative contracts using a risk-neutral binomial model. The pricing process for interest rate derivatives requires intensive calculations and is very tedious. However, the relationship becomes straightforward when it is modeled to support risk neutrality. Understand the concepts of backward induction and how the addition of time steps will increase the accuracy of any bond pricing model. Bonds with embedded options are also discussed in this topic. Be familiar with the price/yield relationship of both callable and putable bonds. This topic incorporates elements of material from the FRM Part I curriculum where you valued options with binomial trees.

INTEREST RATE TREE (BINOMIAL) MODEL

AIM 9.1: Using replicating portfolios, develop and use an arbitrage argument to price a call option on a zero-coupon security. In addition:

* Explain why the option cannot be properly priced using expected discounted values.
* Explain the role of up-state and down-state probabilities in the option valuation.

AIM 9.4: Explain how the principles of arbitrage pricing of derivatives on fixed income securities can be extended over multiple periods.

The binomial interest rate model is used throughout this topic to illustrate the issues that must be considered when valuing bonds with embedded options. A binomial model is a model that assumes that interest rates can take only one of two possible values in the next period.

This interest rate model makes assumptions about interest rate volatility, along with a set of paths that interest rates may follow over time. This set of possible interest rate paths is referred to as an interest rate tree.
Binomial Interest Rate Tree

The diagram in Figure 1 depicts a binomial interest rate tree.

Figure 1: 2-Period Binomial

To understand this 2-period binomial tree, consider the nodes indicated with the boxes in Figure 1. A node is a point in time when interest rates can take one of two possible paths—an upper path, \( U \), or a lower path, \( L \). Now consider the node on the right side of the diagram where the interest rate \( i_{2,LU} \) appears. This is the rate that will occur if the initial rate, \( i_0 \), follows the lower path from node 0 to node 1 to become \( i_{1,L} \), then follows the upper of the two possible paths to node 2, where it takes on the value \( i_{2,LU} \). At the risk of stating the obvious, the upper path from a given node leads to a higher rate than the lower path. Notice also that an upward move followed by a downward move gets us to the same place on the tree as a down-then-up move, so \( i_{2,LU} = i_{2,UL} \).

The interest rates at each node in this interest rate tree are 1-period forward rates corresponding to the nodal period. Beyond the root of the tree, there is more than one 1-period forward rate for each nodal period (i.e., at year 1, we have two 1-year forward rates, \( i_{1,LU} \) and \( i_{1,L} \)). The relationship among the rates associated with each individual nodal period is a function of the interest rate volatility assumption of the model being employed to generate the tree.

Constructing the Binomial Interest Rate Tree

The construction of an interest rate tree, binomial or otherwise, is a tedious process. In practice, the interest rate tree is usually generated using specialized computer software. There is one underlying rule governing the construction of an interest rate tree: *The values for on-the-run issues generated using an interest rate tree should prohibit arbitrage opportunities.* This means that the value of an on-the-run issue produced by the interest rate tree must equal its market price. It should be noted that in accomplishing this, the interest rate tree must maintain the interest rate volatility assumption of the underlying model.

Valuing an Option-Free Bond With the Tree, Using Backward Induction

*Backward induction* refers to the process of valuing a bond using a binomial interest rate tree. The term “backward” is used because in order to determine the value of a bond at node 0, you need to know the values that the bond can take on at node 1. But to determine the values of the bond at node 1, you need to know the possible values of the bond at node 2,
and so on. Thus, for a bond that has \( N \) compounding periods, the current value of the bond is determined by computing the bond's possible values at period \( N \) and working "backward" to node 0.

Consider the binomial tree shown in Figure 2 for a $100 face value, 7.0% annual coupon bond, with two years remaining until maturity, and a market price of $102.999. Starting on the top line, the blocks at each node include the value of the bond, the coupon payment, and the 1-year forward rate at that node. For example, at the upper path of node 1, the price is $99.830, the coupon payment is $7, and the 1-year forward rate is 7.1826%.

**Figure 2: Valuing a 2-Year, 7.0% Coupon, Option-Free Bond**

Know that the value of a bond at a given node in a binomial tree is the average of the present values of the two possible values from the next period. The appropriate discount rate is the forward rate associated with the node under analysis.

**Example: Valuing an option-free bond**

Assuming the bond's market price is $102.99, demonstrate that the tree in Figure 2 is arbitrage free using backward induction.

**Answer:**

Consider the value of the bond at the upper node for period 1, \( V_{1,U} \):

\[
V_{1,U} = \frac{($107 \times 0.5) + ($107 \times 0.5)}{1.071826} = $99.830
\]

Similarly, the value of the bond at the lower node for period 1, \( V_{1,L} \) is:

\[
V_{1,L} = \frac{($107 \times 0.5) + ($107 \times 0.5)}{1.053210} = $101.594
\]
Now calculate $V_0$, the current value of the bond at node 0:

$$V_0 = \frac{\$106.830 \times 0.5 + \$108.594 \times 0.5}{1.045749} = \$102.999$$

Since the computed value of the bond equals the market price, the binomial tree is arbitrage free.

**Professor’s Note:** When valuing bonds, remember to include the coupon payments within your calculation of value. The $106.83, in this example, results from taking the price of the bond at $99.83 and adding the $7 coupon.

**AIM 9.2:** Define risk-neutral pricing and explain how it is used in option pricing.

**AIM 9.3:** Relate the difference between true and risk-neutral probabilities to interest rate drift.

Using the 0.5 probabilities for up and down states as shown in the previous example may not produce an expected discounted value that exactly matches the market price of the bond. This is because the 0.5 probabilities are the assumed true probabilities of price movements. In order to equate the discounted value using a binomial tree and the market price, we need to use what is known as risk-neutral probabilities. Any difference between the risk-neutral and true probabilities is referred to as the interest rate drift.

**Using the Risk-Neutral Interest Rate Tree**

There are actually two ways to compute bond and bond derivative values using a binomial model. These techniques are referred to as risk-neutral pricing.

- The first method is to start with spot and forward rates derived from the current yield curve and then adjust the interest rates on the paths of the tree so that the value derived from the model is equal to the current market price of an on-the-run bond (i.e., the tree is created to be “arbitrage-free”). This is the method we used in the previous example. Once the interest rate tree is derived for an on-the-run bond, we can use it to price derivative securities on the bond by calculating the expected discounted value at each node using the real-world probabilities.

- The second method is to take the rates on the tree as given and then adjust the probabilities so that the value of the bond derived from the model is equal to its current market price. Once we derive these risk-neutral probabilities, we can use them to price derivative securities on the bond by once again calculating the expected discounted value at each node using the risk-neutral probabilities and working backward through the tree.

The value of the derivative is the same under either method.
There are three basic steps to valuing an option on a fixed-income instrument using a binomial tree:

**Step 1:** Price the bond value at each node using the projected interest rates.
**Step 2:** Calculate the intrinsic value of the derivative at each node at maturity.
**Step 3:** Calculate the expected discounted value of the derivative at each node using the risk-neutral probabilities and working backward through the tree.

Note that the option cannot be properly priced using expected discounted values because the call option value depends on the path of interest rates over the life of the option. Incorporating the various interest rate paths will prohibit arbitrage from occurring.

**Example: Call option**

Assume that you want to value a European call option with two years to expiration and a strike price of $100.00. The underlying is a 7%, annual-coupon bond with three years to maturity. Figure 3 represents the first two years of the binomial tree for valuing the underlying bond. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2.

Fill in the missing data in the binomial tree, and calculate the value of the European call option.

*Professor’s Note: Since the option is European, it can only be exercised at maturity.*

**Figure 3: Incomplete Binomial Tree for European Call Option on 3-Year, 7% Bond**
Answer:

Step 1: Calculate the bond prices at each node using the backward induction methodology.

At the middle node in year 2, the price is $100.62. You can calculate this by noting that at the end of year 2 the bond has one year left to maturity:

\[ N = 1; \frac{1}{Y} = 6.34; \ PMT = 7; \ FV = 100; \ CPT \rightarrow PV = 100.62 \]

At the bottom node in year 2, the price is $102.20:

\[ N = 1; \frac{1}{Y} = 4.70; \ PMT = 7; \ FV = 100; \ CPT \rightarrow PV = 102.20 \]

At the top node in year 1, the price is $100.37:

\[ \frac{($105.56 \times 0.6) + ($107.62 \times 0.4)}{1.0599} = 100.37 \]

At the bottom node in year 1, the price is $103.65:

\[ \frac{($107.62 \times 0.6) + ($109.20 \times 0.4)}{1.0444} = 103.65 \]

Today, the price is $105.01:

\[ \frac{($107.37 \times 0.76) + ($110.65 \times 0.24)}{1.03} = 105.01 \]

As shown here, the price at a given node is the expected discounted value of the cash flows associated with the two nodes that "feed" into that node. The discount rate that is applied is the prevailing interest rate at the given node. Note that since this is a European option, you really only need the bond prices at the maturity date of the option (end of year 2) if you are given the arbitrage-free interest rate tree. However, it's good practice to compute all the bond prices.

Step 2: Determine the intrinsic value of the option at maturity in each node. For example, the intrinsic value of the option at the bottom node at the end of year 2 is $2.20 = $102.20 - $100.00. At the top node in year 2, the intrinsic value of the option is zero since the bond price is less than the call price.

Step 3: Using the backward induction methodology, calculate the option value at each node prior to expiration. For example, at the top node for year 1, the option price is $0.23:

\[ \frac{($0.00 \times 0.6) + ($0.62 \times 0.4)}{1.0599} = 0.23 \]
Figure 4 shows the binomial tree with all values included.

**Figure 4: Completed Binomial Tree for European Call Option on 3-Year, 7% Bond**

The option value today is computed as:

\[
\frac{(0.23 \times 0.76) + (1.20 \times 0.24)}{1.03} = 0.45
\]

**Nonrecombining Trees**

AIM 9.5: Describe the rationale behind the use of non-recombining trees in option pricing.

In the previous example, the interest rate in the middle node of period two was the same (i.e., 6.34%) regardless of the path being up then down or down then up. It may be the case, in a practical setting, that the up then down scenario produces a different rate than the down then up scenario. An example of this type of tree may result when any interest rate above a certain level (e.g., 3%) causes rates to move a fixed number of basis points, but any interest rate below that level causes rates to move at a pace that is below the up state’s fixed amount. When rates move in this fashion, the movement process is known as state-dependent volatility, and it results in nonrecombining trees. From an economic standpoint, nonrecombining trees are appropriate; however, prices can be very difficult to calculate when the binomial tree is extended to multiple periods. As a result, practitioners usually avoid implementing these types of binomial trees.
**Constant Maturity Treasury Swap**

AIM 9.6: Calculate the value of a constant maturity Treasury swap, given an interest rate tree and the risk-neutral probabilities.

In addition to valuing options with binomial interest rate trees, we can also value other derivatives such as swaps. The following example calculates the price of a constant maturity Treasury (CMT) swap. A CMT swap is an agreement to swap a floating rate for a Treasury rate such as the 10-year rate.

**Example: CMT Swap**

Assume that you want to value a constant maturity Treasury (CMT) swap. The swap pays the following every six months until maturity:

\[
\left( \frac{\$1,000,000}{2} \right) \times \left( y_{CMT} - 7\% \right)
\]

\( y_{CMT} \) is a semiannually compounded yield, of a predetermined maturity, at the time of payment (\( y_{CMT} \) is equivalent to 6-month spot rates). Assume there is a 76% risk-neutral probability of an increase in the 6-month spot rate and a 60% risk-neutral probability of an increase in the 1-year spot rate.

Fill in the missing data in the binomial tree, and calculate the value of the swap.

**Figure 5: Incomplete Binomial Tree for CMT Swap**
Answer:

In six months, the top node and bottom node payoffs are, respectively:

\[ \text{payoff}_{1,U} = \frac{\$1,000,000}{2} \times (7.25\% - 7.00\%) = \$1,250 \]

\[ \text{payoff}_{1,L} = \frac{\$1,000,000}{2} \times (6.75\% - 7.00\%) = -\$1,250 \]

Similarly in one year, the top, middle, and bottom payoffs are, respectively:

\[ \text{payoff}_{2,U} = \frac{\$1,000,000}{2} \times (7.50\% - 7.00\%) = \$2,500 \]

\[ \text{payoff}_{2,M} = \frac{\$1,000,000}{2} \times (7.00\% - 7.00\%) = \$0 \]

\[ \text{payoff}_{2,L} = \frac{\$1,000,000}{2} \times (6.50\% - 7.00\%) = -\$2,500 \]

The possible prices in six months are given by the expected discounted value of the 1-year payoffs under the risk-neutral probabilities, plus the 6-month payoffs ($1,250 and $-1,250). Hence, the 6-month values for the top and bottom node are, respectively:

\[ V_{1,U} = \frac{\left(\$2,500 \times 0.6\right) + \left(\$0 \times 0.4\right)}{1 + 0.0725} + \$1,250 = \$2,697.53 \]

\[ V_{1,L} = \frac{\left(\$0 \times 0.6\right) + \left(-\$2,500 \times 0.4\right)}{1 + 0.0675} - \$1,250 = -\$2,217.35 \]

Today the price is $1,466.63, calculated as follows:

\[ V_0 = \frac{\left(\$2,697.53 \times 0.76\right) + \left(-\$2,217.35 \times 0.24\right)}{1 + 0.07} = \$1,466.63 \]
Figure 6 shows the binomial tree with all values included.

**Figure 6: Completed Binomial Tree for CMT Swap**

![Diagram of the binomial tree showing interest rates and swap prices at different time steps.]

**Time Steps**

AIM 9.7: Discuss the advantages and disadvantages of reducing the size of the time steps on the pricing of derivatives on fixed income securities.

For the sake of simplicity, the previous example assumed periods of six months. However, in reality, the time between steps should be much smaller. As you can imagine, the smaller the time between steps, the more complicated the tree and calculations become. Using daily time steps will greatly enhance the accuracy of any model but at the expense of additional computational complexity.

**Fixed-Income Securities and Black-Scholes-Merton**

AIM 9.8: Explain why the Black-Scholes-Merton model is not appropriate to value derivatives on fixed-income securities.

The Black-Scholes-Merton model is the most well-known equity option-pricing model. Unfortunately, the model is based on three assumptions that do not apply to fixed-income securities:

1. The model's main shortcoming is that it assumes there is no upper limit to the price of the underlying asset. However, bond prices do have a maximum value. This upper limit occurs when interest rates equal zero so that zero-coupon bonds are priced at par and coupon bonds are priced at the sum of the coupon payments plus par.
2. It assumes the risk-free rate is constant. However, changes in short-term rates do occur, and these changes cause rates along the yield curve and bond prices to change.

3. It assumes bond price volatility is constant. With bonds, however, price volatility decreases as the bond approaches maturity.

**Bonds With Embedded Options**

AIM 9.9: Describe the impact of embedded options on the value of fixed-income securities.

Fixed-income securities are often issued with embedded options, such as a call feature. In this case, the price/yield relationship will change, and so will the price volatility characteristics of the issue.

**Callable Bonds**

A call option gives the issuer the right to buy back the bond at fixed prices at one or more points in the future, prior to the date of maturity. Since the investor takes a short position in the call, the right to purchase rests with the issuer. Such bonds are deemed to be callable (note that a call provision on a bond is analytically similar to a prepayment option).

**Figure 7: Price-Yield Function of Callable Bond**

For an option-free noncallable bond, prices will fall as yields rise, and prices will rise unabated as yields fall—in other words, they’ll move in line with yields. That’s not the case, however, with callable bonds. As you can see in Figure 7, the decline in callable bond yield will reach the point where the rate of increase in the price of the callable bond will start slowing down and eventually level off.

This is known as negative convexity. Such behavior is due to the fact that the issuer has the right to retire the bond prior to maturity at some specified call price. The call price, in effect, acts to hold down the price of the bond (as rates fall) and causes the price/yield curve to flatten. The point where the curve starts to flatten is at (or near) a yield level of \( y' \).
Note that as long as yields remain above \( y' \), a callable bond will behave like any option-free (noncallable) issue and exhibit positive convexity. That's because at high yield levels, there is little chance of the bond being called.

Below \( y' \), investors begin to anticipate that the firm may call the bond, in which case investors will receive the call price. Therefore, as yield levels drop, the bond’s market value is bounded from above by the call price. Thus, callability effectively caps the investor’s capital gains as yields fall. Moreover, it exacerbates reinvestment risk since it increases the cash flow that must be reinvested at lower rates (i.e., without the call or prepayment option, the cash flow will only be the coupon; with the option, the cash flow is the coupon plus the call price).

Thus, in Figure 7, as long as yields remain below \( y' \), callable bonds will exhibit price compression, or negative convexity; however, at yields above \( y' \), those same callable bonds will exhibit all the properties of positive convexity.

**Putable Bonds**

The put feature in putable bonds is another type of embedded option. The put feature gives the bondholder the right to sell the bond back to the issuer at a set price (i.e., the bondholder can “put” the bond to the issuer). The impact of the put feature on the price/yield relationship is shown in Figure 8.

**Figure 8: Price-Yield Function of a Putable Bond**

![Diagram of Price-Yield Function of a Putable Bond]

At low yield levels relative to the coupon rate, the price/yield relationship of putable and nonputable bonds is similar. However, as shown in Figure 8, if yields rise above \( y' \), the price of the putable bond does not fall as rapidly as the price of the option-free bond. This is because the put price serves as a floor value for the price of the bond.
KEY CONCEPTS

1. Risk-neutral, or no-arbitrage, binomial tree models are used to allow for proper valuation of bonds with embedded options.

2. Callable bonds can be valued by modifying the cash flows at each node in the interest rate tree to reflect the cash flow prescribed by the embedded call option according to the call rule.

3. Backward-induction methodology with a binomial model requires discounting of the cash flows that occur at each node in an interest rate tree (bond value plus coupon payment) backward to the root of the tree. For bonds with one or more embedded options, the bond value that must be discounted at each node depends on whether the embedded option will be exercised.

4. The precision of a model can be improved by reducing the length of the time steps, but the trade-off is increased complexity.

5. The Black-Scholes-Merton model cannot be used for the valuation of fixed-income securities because it makes the following unreasonable assumptions:
   - There is no upper price bound.
   - The risk-free rate is constant.
   - Bond volatility is constant.

6. Fixed-income securities are often issued with embedded options. When embedded options are present, the price/yield relationship will change, and so will the price volatility characteristics of the issue.
1. A European put option has two years to expiration and a strike price of $101.00. The underlying is a 7% annual coupon bond with three years to maturity. Assume that the risk-neutral probability of an up move is 0.76 in year 1 and 0.60 in year 2. The current interest rate is 3.00%. At the end of year 1, the rate will either be 5.99% or 4.44%. If the rate in year 1 is 5.99%, it will either rise to 8.56% or rise to 6.34% in year 2. If the rate in one year is 4.44%, it will either rise to 6.34% or rise to 4.70%. The value of the put option today is closest to:
   A. $1.17.
   B. $1.30.
   C. $1.49.
   D. $1.98.

2. The Black-Scholes-Merton option pricing model is not appropriate for valuing options on corporate bonds because corporate bonds:
   A. have credit risk.
   B. have an upper price bound.
   C. have constant price volatility.
   D. are not priced by arbitrage.

3. Which of the following regarding the use of small time steps in the binomial model is true?
   A. Less realistic model.
   B. More accurate model.
   C. Less complicated computations.
   D. Less computational expense.

4. Which of the following statements about callable bonds compared to noncallable bonds is false?
   A. They have less price volatility.
   B. They have negative convexity.
   C. Capital gains are capped as yields rise.
   D. At low yields, reinvestment rate risk rises.

5. Which of the following statements concerning the calculation of value at a node in a fixed income binomial interest rate tree is most accurate? The value at each node is the:
   A. present value of the two possible values from the next period.
   B. average of the present values of the two possible values from the next period.
   C. sum of the present values of the two possible values from the next period.
   D. average of the future values of the two possible values from the next period.
CONCEPT CHECKER ANSWERS

1. A This is the same underlying bond and interest rate tree as in the call option example from this topic. However, here we are valuing a put option.

The option value in the upper node at the end of year 1 is computed as:

\[
\frac{(2.44 \times 0.6) + (0.38 \times 0.4)}{1.0599} = 1.52
\]

The option value in the lower node at the end of year 1 is computed as:

\[
\frac{(0.38 \times 0.6) + (0.00 \times 0.4)}{1.0444} = 0.22
\]

The option value today is computed as:

\[
\frac{(1.52 \times 0.76) + (0.22 \times 0.24)}{1.0300} = 1.17
\]

2. B The Black-Scholes-Merton model cannot be used for the valuation of fixed-income securities because it makes the following assumptions, which are not reasonable for valuing fixed-income securities:
   • There is no upper price bound.
   • The risk-free rate is constant.
   • Bond volatility is constant.

3. B The use of small time steps in the binomial model yields a more realistic model, a more accurate model, more complicated computations, and more computational expense.

4. C Callable bonds have the following characteristics:
   • Lower price volatility.
   • Negative convexity.
   • Capital gains are capped as yields fall.
   • Exhibit increased reinvestment rate risk when yields fall.

5. B The value at any given node in a binomial tree is the average present value of the cash flows at the two possible states immediately to the right of the given node, discounted at the 1-period rate at the node under examination.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**VOLATILITY SMILES**

**Exam Focus**

This topic discusses some of the reasons for the existence of volatility smiles, and how volatility affects option pricing as well as other option characteristics. Focus on the explanation of why volatility smiles exist in currency and equity options. Also, understand how volatility smiles impact the Greeks and how to interpret price jumps.

**Put-Call Parity**

AIM 10.2: Explain how put-call parity indicates that the implied volatility used to price call options is the same used to price put options.

Recall that put-call parity is a no-arbitrage equilibrium relationship that relates European call and put option prices to the underlying asset’s price and the present value of the option’s strike price. In its simplest form, put-call parity can be represented by the following relationship:

\[ c - p = S - PV(X) \]

where:
- \( c \) = price of a call
- \( p \) = price of a put
- \( S \) = price of the underlying security
- \( PV(X) \) = present value of the strike

\( PV(X) \) can be represented in continuous time by:

\[ PV(X) = Xe^{-rT} \]

where:
- \( r \) = risk-free rate
- \( T \) = time left to expiration expressed in years

Since put-call parity is a no-arbitrage relationship, it will hold whether or not the underlying asset price distribution is lognormal, as required by the Black-Scholes-Merton (BSM) option pricing model.
If we simply rearrange put-call parity and denote subscripts for the option prices to indicate whether they are market or Black-Scholes-Merton option prices, the following two equations are generated:

\[ P_{\text{mkt}} + S_0 e^{-qt} = c_{\text{mkt}} + PV(X) \]
\[ P_{\text{BSM}} + S_0 e^{-qt} = c_{\text{BSM}} + PV(X) \]

Subtracting the second equation from the first gives us:

\[ P_{\text{mkt}} - P_{\text{BSM}} = c_{\text{mkt}} - c_{\text{BSM}} \]

This relationship shows that, given the same strike price and time to expiration, option market prices that deviate from those dictated by the Black-Scholes-Merton model are going to deviate in the same amount whether they are for calls or puts. Since any deviation in prices will be the same, the implication is that the implied volatility of a call and put will be equal for the same strike price and time to expiration.

**Volatility Smiles**

AIM 10.1: Define volatility smile and volatility skew.

Actual option prices, in conjunction with the BSM model, can be used to generate implied volatilities which may differ from historical volatilities. When option traders allow implied volatility to depend on strike price, patterns of implied volatility are generated which resemble “volatility smiles.” These curves display implied volatility as a function of the option's strike (or exercise) price. In this topic, we will examine volatility smiles for both currency and equity options. In the case of equity options, the volatility smile is sometimes referred to as a volatility skew since, as we will see in AIM 10.5, the volatility pattern for equity options is essentially an inverse relationship.

**Currency Options**

AIM 10.3: Relate the shape of the volatility smile (or skew) to the shape of the implied distribution of the underlying asset price and to the pricing of options on the underlying asset.

AIM 10.4: Explain why foreign exchange rates are not necessarily lognormally distributed and the implications this can have on option prices and implied volatility.

The volatility pattern used by traders to price currency options generates implied volatilities that are higher for deep in-the-money and deep out-of-the-money options, as compared to the implied volatility for at-the-money options, as shown in Figure 1.
The easiest way to see why implied volatilities for away-from-the-money options are greater than at-the-money options is to consider the following call and put examples. For calls, a currency option is going to pay off only if the actual exchange rate is above the strike rate. For puts, on the other hand, a currency option is going to pay off only if the actual exchange rate is below the strike rate. If the implied volatilities for actual currency options are greater for away-from-the-money than at-the-money options, currency traders must think there is a greater chance of extreme price movements than predicted by a lognormal distribution. Empirical evidence indicates that this is the case.

This tendency for exchange rate changes to be more extreme is a function of the fact that exchange rate volatility is not constant and frequently jumps from one level to another, which increases the likelihood of extreme currency rate levels. However, these two effects tend to be mitigated for long-dated options, which tend to exhibit less of a volatility smile pattern than shorter-dated options.

**Equity Options**

AIM 10.5: Discuss the volatility smile for equity options and give possible explanations for its shape.

The equity option volatility smile is different from the currency option pattern. The smile is more of a “smirk,” or skew, that shows a higher implied volatility for low strike price options (in-the-money calls and out-of-the-money puts) than for high strike price options (in-the-money puts and out-of-the-money calls). As shown in Figure 2, there is essentially an inverse relationship between implied volatility and the strike price of equity options.
The volatility smirk (half-smile) exhibited by equity options translates into a left-skewed implied distribution of equity price changes. This left-skewed distribution indicates traders believe the probability of large downward movements in price is greater than large upward movements in price, as compared with a lognormal distribution. Two reasons have been promoted as causing this increased implied volatility—leverage and "crashophobia."

- **Leverage.** When a firm's equity value decreases, the amount of leverage increases, which essentially increases the riskiness, or "volatility," of the underlying asset. When a firm's equity increases in value, the amount of leverage decreases, which tends to decrease the riskiness of the firm. This lowers the volatility of the underlying asset. All else held constant, there is an inverse relationship between volatility and the underlying asset's valuation.

- **Crashophobia.** The second explanation, used since the 1987 stock market crash, was coined "crashophobia" by Mark Rubinstein. Market participants are simply afraid of another market crash, so they place a premium on the probability of stock prices falling precipitously—deep out-of-the-money puts will exhibit high premiums since they provide protection against a substantial drop in equity prices. There is some support for Rubinstein's crashophobia hypothesis, because the volatility skew tends to increase when equity markets decline, but is not as noticeable when equity markets increase in value.

### Alternative Methods for Studying Volatility Smiles

AIM 10.6: Describe alternative ways of characterizing the volatility smile.

The volatility smiles we have characterized thus far have examined the relationship between implied volatility and strike price. Other relationships exist which allow traders to use alternative methods to study these volatility patterns. All alternatives require a replacement of the independent variable, strike price (X).

One alternative method involves replacing the strike price with strike price divided by stock price (X / S₀). This method results in a more stable volatility smile. A second alternative approach is to substitute the strike price with strike price divided by the forward price for the underlying asset (X / F₀). The forward price would have the same maturity date as the options being assessed. Traders sometimes view the forward price as a better gauge of at-the-money option prices since the forward price displays the theoretical expected stock price.
A third alternative method involves replacing the strike price with the option's delta. With this approach, traders are able to study volatility smiles of options other than European and American options.

**Volatility Term Structure and Volatility Surfaces**

AIM 10.7: Describe volatility term structures and volatility surfaces and how they may be used to price options.

The volatility term structure is a listing of implied volatilities as a function of time to expiration for at-the-money option contracts. When short-dated volatilities are low (from historical perspectives), volatility tends to be an increasing function of maturity. When short-dated volatilities are high, volatility tends to be an inverse function of maturity. This phenomenon is related to, but has a slightly different meaning from, the mean-reverting characteristic often exhibited by implied volatility.

A volatility surface is nothing other than a combination of a volatility term structure with volatility smiles (i.e., those implied volatilities away-from-the-money). The surface provides guidance in pricing options with any strike or maturity structure.

A trader's primary objective is to maintain a pricing mechanism that generates option prices consistent with market pricing. Even if the implied volatility or model pricing errors change due to shifting from one pricing model to another (which could occur if traders use an alternative model to Black-Scholes-Merton), the objective is to have consistency in model-generated pricing. The volatility term structure and volatility surfaces can be used to confirm or disprove a model's accuracy and consistency in pricing.

**The Option Greeks**

AIM 10.8: Explain the impact of the volatility smile on the calculation of the "Greeks."

Option Greeks indicate expected changes in option prices given changes in the underlying factors that affect option prices.

The problem here is that option Greeks, including delta and vega, may be affected by the implied volatility of an option. Remember these guidelines for how implied volatility may affect the Greek calculations of an option:

- The first guideline is the sticky strike rule, which makes an assumption that an option's implied volatility is the same over short time periods (e.g., successive days). If this is the case, the Greek calculations of an option are assumed to be unaffected, as long as the implied volatility is unchanged. If implied volatility changes, the option sensitivity calculations may not yield the correct figures.
The second guideline is the sticky delta rule, which assumes the relationship between an option’s price and the ratio of underlying to strike price applies in subsequent periods. The idea here is that the implied volatility reflects the moneyness of the option, so the delta calculation includes an adjustment factor for implied volatility. If the sticky delta rule holds, the option’s delta will be larger than that given by the Black-Scholes-Merton formula.

Keep in mind, however, that both rules assume the volatility smile is flat for all option maturities. If this is not the case, the rules are not internally consistent and, to correct for a non-flat volatility smile, we would have to rely on an implied volatility function or tree to correctly calculate option Greeks.

**PRICE JUMPS**

AIM 10.9: Explain the impact of asset price jumps on volatility smiles.

Price jumps can occur for a number of reasons. One reason may be the expectation of a significant news event that causes the underlying asset to move either up or down by a large amount. This would cause the underlying distribution to become bimodal, but with the same expected return and standard deviation as a unimodal, or standard, price-change distribution.

Implied volatility is affected by price jumps and the probabilities assumed for either a large up or down movement. The usual result, however, is that at-the-money options tend to have a higher implied volatility than either out-of-the-money or in-the-money options. Away-from-the-money options exhibit a lower implied volatility than at-the-money options. Instead of a volatility smile, price jumps would generate a volatility frown, as in Figure 3.

**Figure 3: Volatility Smile (Frown) With Price Jump**

![Volatility Smile (Frown) With Price Jump](image)
KEY CONCEPTS

1. Put-call parity indicates that the deviation between market prices and Black-Scholes-Merton prices will be equivalent for calls and puts. Hence, implied volatility will be the same for calls and puts.

2. Empirical evidence tends to indicate that away-from-the-money currency options generate volatility smiles due to the expectation that large movements of exchange rates will occur more often than is predicted by a lognormal distribution.

3. The volatility smile exhibited by equity options is more of a “smirk,” with implied volatility higher for low strike prices. This has been attributed to leverage and “crashophobia” effects.

4. Volatility term structures and volatility surfaces are used by traders to judge consistency in model-generated option prices.

5. Volatility smiles that are not flat require the use of implied volatility functions or trees to correctly calculate option Greeks.

6. Price jumps may generate volatility “frowns” instead of smiles.
1. The market price deviations for puts and calls from Black-Scholes-Merton prices indicate:
   A. equivalent put and call implied volatility.
   B. equivalent put and call moneyness.
   C. unequal put and call implied volatility.
   D. unequal put and call moneyness.

2. An empirical distribution that exhibits a fatter right tail than that of a lognormal distribution would indicate:
   A. equal implied volatilities across low and high strike prices.
   B. greater implied volatilities for low strike prices.
   C. greater implied volatilities for high strike prices.
   D. higher implied volatilities for mid-range strike prices.

3. The “sticky strike rule” assumes that implied volatility is:
   A. the same across maturities for given strike prices.
   B. the same for short time periods.
   C. the same across strike prices for given maturities.
   D. different across strike prices for given maturities.

4. Compared to at-the-money currency options, out-of-the-money currency options exhibit which of the following volatility traits?
   A. Lower implied volatility.
   B. A frown.
   C. A smirk.
   D. Higher implied volatility.

5. Which of the following regarding equity option volatility is true?
   A. There is higher implied price volatility for away-from-the-money equity options.
   B. “Crashophobia” suggests actual equity volatility increases when stock prices decline.
   C. Compared to the lognormal distribution, traders believe the probability of large down movements in price is similar to large up movements.
   D. Increasing leverage at lower equity prices suggests increasing volatility.
CONCEPT CHECKER ANSWERS

1. A Put-call parity indicates that the implied volatility of a call and put will be equal for the same strike price and time to expiration.

2. C An empirical distribution with a fat right tail generates a higher implied volatility for higher strike prices due to the increased probability of observing high underlying asset prices. The pricing indication is that in-the-money calls and out-of-the-money puts would be "expensive."

3. B The sticky strike rule, when applied to calculating option sensitivity measures, assumes implied volatility is the same over short time periods.

4. D Away-from-the-money currency options have greater implied volatility than at-the-money options. This pattern results in a volatility smile.

5. D There is higher implied price volatility for low strike price equity options. "Crashophobia" is based on the idea that large price declines are more likely than assumed in Black-Scholes-Merton prices, not that volatility increases when prices decline. Compared to the lognormal distribution, traders believe the probability of large down movements in price is higher than large up movements. Increasing leverage at lower equity prices suggests increasing volatility.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**Exotic Options**

**Exam Focus**

In this topic, we define and discuss the important characteristics of a variety of exotic options. The difference between exotic options and more traditional exchange-traded instruments is also highlighted. Be familiar with the payoff structures for the various exotic options discussed.

**Evaluating Exotic Options**

AIM 11.1: Define and contrast exotic derivatives and plain vanilla derivatives.

AIM 11.2: Describe some of the factors that drive the development of exotic products.

Plain vanilla derivatives include listed futures contracts and commonly used forwards and other over-the-counter (OTC) derivatives that are traded in fairly liquid markets. Exotic derivatives are customized to fit a specific firm need for hedging that cannot be met by plain vanilla derivatives. With plain vanilla derivatives, there is little uncertainty about the cost, the current market value, when they will pay, how much they will pay, and the cost of exiting the position. With exotic derivatives, some or all of these may be in question.

Exotic derivatives are developed for several reasons. The main purpose is to provide a unique hedge for a firm’s underlying assets. Other reasons include addressing tax and regulatory concerns as well as speculating on the expected future direction of market prices.

Four questions that should be considered when evaluating exotic derivative strategies are:

- Will the strategy pay in the right circumstances to provide an effective hedge? Problems with understanding the payoff of the exotic derivative and credit risk of the derivative strategy can lead to a difference between the payoff the user expects and the actual payoff received.
- What is the cost of the exotic derivative hedging strategy?
- Is a pricing model needed, and does the user have the appropriate pricing model to estimate dealer cost and monitor the value of non-traded derivatives over time?
- How is a derivative position reversed? Note that the costs of exiting a position or strategy may involve penalties and large bid-ask spreads or require a pricing model to evaluate alternatives.
**Using Packages to Formulate a Zero-Cost Product**

AIM 11.3: Explain how any derivative can be converted into a zero-cost product.

A package is defined as some combination of standard European options, forwards, cash, and the underlying asset. Bull, bear, and calendar spreads, as well as straddles and strangles, are examples of packages. Packages usually consist of selling one instrument with certain characteristics and buying another with somewhat different characteristics. Because packages often consist of a long position and a short position, they can be constructed so that the initial cost to the investor is zero.

For example, consider a zero-cost short collar. A short collar combines a long standard put option with an exercise price \( X_L \) and a short standard call option with exercise price \( X_H \) (where \( X_L < X_H \)). If the premium the investor pays for the put option is exactly offset by the premium the investor receives for the short call position, the investor's net cost for implementing the short collar strategy is zero. In any case where the investor's cash outflows from long positions are offset by cash inflows from short positions, the investor can use a package to create a zero-cost product.

**Transforming Standard American Options into Nonstandard American Options**

AIM 11.4: List and describe how various option characteristics can transform standard American options into nonstandard American options.

Recall from the FRM Part I curriculum that standard exchange-traded American options can be exercised at any time prior to expiration. If some of the available expiration periods are restricted, or changes are made to other standard features, standard options become what we refer to as nonstandard options. Nonstandard options are common in the over-the-counter (OTC) market.

There are three common features that transform standard American options into nonstandard options:

- The most common transformation can be made to restrict early exercise to certain dates (e.g., a three-month call option may only be exercised on the last day of each month.) This type of transformation results in a **Bermudan option**.
- Early exercise can be limited to a certain portion of the life of the option (e.g., there is a "lock out" period that does not allow a 6-month call option to be exercised in the first three months of the call's life).
- The option's strike price may change (e.g., the strike price of a 3-year call option with a strike price of 40 at initiation may rise to 44 in year 2 and 48 in year 3).
EXOTIC OPTION PAYOFF STRUCTURES

AIM 11.5: List and describe the characteristics and pay-off structure of:

- Forward start options
- Compound options
- Chooser and barrier options
- Binary options
- Lookback options
- Shout options
- Asian options
- Exchange options
- Rainbow options
- Basket options

Forward Start Options

Forward start options are options that begin their existence at some time in the future. For example, today an investor may purchase a 3-month call option that will not come into existence until six months from today. Employee incentive plans commonly incorporate forward start options in which at-the-money options will be created after some period of employment has passed. Note that when the underlying asset is a nondividend paying stock, the value of a forward start option will be identical to the value of a European at-the-money option with the same time to expiration as the forward start option.

Compound Options

Compound options are options on options. There are four key types of compound options:

- A call on a call gives the investor the right to buy a call option at a set price for a set period of time.
- A call on a put gives the investor the right to buy a put option at a set price for a set period of time.
- A put on a call gives the investor the right to sell a call option at a set price for a set period of time.
- A put on a put gives the investor the right to sell a put option at a set price for a set period of time.

Compound options have two levels of the underlying that determine their value—the value of the underlying option, which in turn is determined by the value of the underlying asset.

Compound options consist of two strike prices and two exercise dates. The first strike price and exercise date are used by the holder to evaluate whether to exercise the first option to receive the second option, where the second option is an option on the underlying asset, or just let the compound option expire. For example, a call on a call would be exercised if the price of the call on the underlying for the second call option were greater than the strike price of the initial option. The strike price and exercise date on the second call, however, are related to the value of the underlying asset.
Chooser Options

This interesting option allows the owner, after a certain period of time has elapsed, to choose whether the option is a call or a put. The option with the greater value after the requisite time has elapsed will determine whether the owner will choose the option to be a put or a call.

Barrier Options

Barrier options are options whose payoffs (and existence) depend on whether the underlying's asset price reaches a certain barrier level over the life of the option. These options are usually less expensive than standard options, and essentially come in either knock-out or knock-in flavors. Specific types of barrier options are:

- **Down-and-out call (put).** A standard call (put) option that ceases to exist if the underlying asset price hits the barrier level, which is set below the current stock value.
- **Down-and-in call (put).** A standard call (put) option that only comes into existence if the underlying asset price hits the barrier level, which is set below the current stock value.
- **Up-and-out call (put).** A standard call (put) option that ceases to exist if the underlying asset price hits a barrier level, which is set above the current stock value.
- **Up-and-in call (put).** A standard call (put) option that only comes into existence if the underlying asset price hits the above-current stock-price barrier level.

Barrier options have characteristics that can be very different from those of standard options. For example, vega, the sensitivity of an option's price to changes in volatility, is always positive for a standard option but may be negative for a barrier option. Increased volatility on a down-and-out option and an up-and-out option does not increase value because the closer the underlying gets to the barrier price, the greater the chance the option will expire.

Binary Options

Binary options generate discontinuous payoff profiles because they pay only one price at expiration if the asset value is above the strike price. The term binary means that the option payoff has one of two states: the option pays a set dollar amount at expiration if the option is above the strike price, or the option pays nothing if the price is below the strike price. Hence, a payoff discontinuity results from the fact that the payoff is only one value—it does not increase continuously with the price of the underlying asset as in the case of a traditional option.

In the case of a **cash-or-nothing call**, a fixed amount, Q, is paid if the asset ends up above the strike price. Since the Black-Scholes-Merton formula denotes \( N(d_2) \) as the probability of the asset price being above the strike price, the value of a cash-or-nothing call is equal to \( Qe^{-rT}N(d_2) \).

An **asset-or-nothing call** pays the value of the stock when the contract is initiated if the stock price ends up above the strike price at expiration. The corresponding value for this option is \( S_0e^{-rT}N(d_1) \).
Lookback Options

Lookback options are options whose payoffs depend on the maximum or minimum price of the underlying asset during the life of the option. A floating lookback call pays the difference between the expiration price and the minimum price of the stock over the horizon of the option. This essentially allows the owner to purchase the security at its lowest price over the option's life. On the other hand, a floating lookback put pays the difference between the expiration and maximum price of the stock over the time period of the option. This translates into allowing the owner of the option to sell the security at its highest price over the life of the option.

Lookback options can also be fixed when an exercise price is specified. A fixed lookback call has a payoff function that is identical to a European call option. However, for this exotic option, the final stock price (or expiration price) in the European call option payoff is replaced by the maximum price during the option's life. Similarly, a fixed lookback put has a payoff like a European put option but replaces the final stock price with the minimum price during the option's life.

Shout Options

A shout option allows the owner to pick a date when he “shouts” to the option seller, which then translates into an intrinsic value of the option at the time of the shout. At option expiration, the owner receives the maximum of the shout intrinsic value or the option expiration intrinsic value. In other words, for a shout call option, even if the price of the stock falls after the shout, the investor has locked in the difference between the price of the stock and the shout price. If the stock continues to rise, the shout option will have a payoff consistent with a standard call option. Note that most shout options allow for one “shout” during the option’s life.

Asian Options

Asian options have payoff profiles based on the average price of the security over the life of the option. Average price calls and puts pay off the difference between the average stock price and the strike price. Note that the average price will be much less volatile than the actual price. This means that the price for an Asian average price option will be lower than the price of a comparable standard option. Average strike calls and average strike puts pay off the difference between the stock expiration price and average price, which essentially represents the strike price in a typical intrinsic value calculation. If the average price or strike price for an Asian option is based on a geometric average, then using an option pricing model is not a problem because a geometric average is lognormal. However, most Asian options base their average calculations on arithmetic averages, which complicates the pricing process. In this case, a lognormal distribution of prices is assumed, which provides an adequate approximation.
Exchange Options

A common use of an option to exchange one asset for another, often called an exchange option, is to exchange one currency with another. For example, consider a U.S. investor who holds an option to purchase euros with yen at a specified exchange rate. In this particular case, the option will be exercised if euros are more valuable to the U.S. investor than yen. Other applications, such as tender offers to exchange one stock for another, also arise in certain situations.

Basket Options

Basket options are simply options to purchase or sell baskets of securities. These baskets may be defined specifically for the individual investor and may be composed of specific stocks, indices, or currencies. Any exotic options which involve several different assets are more generally referred to as rainbow options.

Volatility and Variance Swaps

AIM 11.6: Describe and contrast volatility and variance swaps.

A volatility swap involves the exchange of volatility based on a notional principal. One side of the swap pays based on a pre-specified fixed volatility while the other side pays based on realized volatility. Unlike the exotic options we have discussed thus far, volatility swaps are a bet on volatility alone as opposed to a bet on volatility and the price of the underlying asset.

Much like a volatility swap, a variance swap involves exchanging a pre-specified fixed variance rate for a realized variance rate. The variance rate being exchanged is simply the square of the volatility rate. However, unlike volatility swaps, variance swaps are easier to price and hedge since they can be replicated using a collection of call and put options.

ISSUES IN HEDGING EXOTIC OPTIONS

AIM 11.7: Explain the basic premise of static option replication and how it can be applied to hedging exotic options.

The typical dynamic option-hedging situation uses option Greeks to measure sensitivity of the option value to changes in underlying asset characteristics (i.e., creating a delta-neutral portfolio). Hedging is simpler with some exotic options than it is with plain vanilla options. Asian options, for instance, depend on the average price of the underlying. Through time, the uncertainty of the average value gets smaller. Hence, the option begins to become less sensitive to changes in the value of the security because the payoff can be estimated more accurately.

Hedging positions in barrier and other exotic options are not so straightforward. This type of hedging requires the replication of a portfolio that is exactly opposite to the option position. When the replication portfolio requires frequent adjustments to the
holdings in the underlying assets, the hedging procedure is referred to as dynamic options replication. **Dynamic options replication** requires frequent trading, which makes it costly to implement.

As an alternative, a **static options replication** approach may be used to hedge positions in exotic options. In this case, a short portfolio of actively traded options that approximates the option position to be hedged is constructed. This short replication options portfolio is created once, which drastically reduces the transaction costs associated with dynamic rebalancing.
**Key Concepts**

1. Packages are portfolios of European options, forwards, cash, and the underlying asset.

2. Restricting exercise dates and changing strike prices can transform standard options into nonstandard options.

3. Forward start options are options that commence in the future.

4. A compound option is defined as an option on another option.

5. Chooser options allow the owner to choose whether the option is a call or a put, after option initiation.

6. Barrier options are options whose payoffs (and existence) depend on whether the underlying’s asset price reaches a certain barrier level over the life of the option.

7. Binary options either pay nothing (if price is below strike price) or a fixed amount at expiration.

8. Lookback options depend on the maximum or minimum value of the underlying asset during the life of the option.

9. Shout options allow the owner to receive either the intrinsic value of the option at the shout date or at expiration, whichever is greater.

10. Asian options have payoff profiles that depend on the average underlying asset price over the life of the option.

11. An exchange option is an option to exchange one asset for another.

12. Basket options allow the owner to buy or sell portfolios of assets.

13. Exotic options can be hedged in either a dynamic or static context, depending on the characteristics of the option.
1. A down-and-in call option is an option that comes into existence only when the underlying asset price:
   A. rises to a set barrier level.
   B. falls to a set barrier level.
   C. falls to a set average barrier level.
   D. rises to a set average barrier level.

2. A cash-or-nothing put option has a payout profile equivalent to zero or:
   A. the underlying asset price if the value of the asset ends below the strike price.
   B. the underlying asset price if the value of the asset ends above the strike price.
   C. a set amount if the value of the asset ends below the strike price.
   D. a set amount if the value of the asset ends above the strike price.

3. An Asian option can be hedged dynamically because the:
   A. average value of the underlying asset price decreases uncertainty the closer the option gets to expiration.
   B. average value of the underlying asset price increases uncertainty the closer the option gets to expiration.
   C. maximum value of the underlying asset price decreases uncertainty the closer the option gets to expiration.
   D. minimum value of the underlying asset price increases uncertainty the closer the option gets to expiration.

4. Which of the following options is most likely to have a negative vega?
   A. A chooser option close to expiration.
   B. A forward start put option before the start date.
   C. An Asian put option close to the beginning of the option’s life.
   D. An up and out put when the stock price is close to the barrier.

5. Under which of the following circumstances would the value of an up and out call option be zero?
   A. The strike price is above the barrier price.
   B. The stock price is below the barrier price.
   C. The stock price is above the strike price.
   D. The stock price is below the strike price.
1. B Down-and-in call options are standard options that come into existence only if the asset price falls to a set barrier price level, which is set below the current stock price.

2. C Cash-or-nothing put options pay only a set amount if the stock price ends below the strike price. These options differ from standard put options because the payment is a set amount that does not continuously increase with the decrease in stock price.

3. A Dynamic hedging can be used to hedge Asian options because uncertainty in the expiration value is decreased the closer one gets to expiration. This occurs because the intrinsic value becomes “set” due to the averaging effect over the life of the option.

4. D Vega is the sensitivity of the price of an option to changes in volatility of the underlying stock. For most options, vega is always positive—as volatility of the underlying stock increases, the price of the option also increases. An exception would be a knockout barrier option when the stock price is close to the barrier. Higher volatility means the barrier is more likely to be reached and the option will cease to exist.

5. A With an up and out call, if the stock price rises beyond the barrier price, the option ceases to exist. It therefore follows that if the strike price is above the barrier price, the option will never come into the money because the option will cease to exist before the option will ever come into the money.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

OVERVIEW OF MORTGAGES AND THE CONSUMER MORTGAGE MARKET

Exam Focus

This is the first of four topics related to mortgages and mortgage-backed securities (MBS). In this topic, we introduce characteristics of mortgage loans, discuss the major players in this market, and address the various risks associated with mortgages and MBS. For the exam, you should have a solid understanding of the eight key attributes that define mortgages. Specific knowledge of the amortization types is a prerequisite to understanding the allocation of interest and principal amounts for the various mortgage types. Finally, an understanding of the issues related to prepayment risk is crucial.

Key Attributes That Define Mortgages

AIM 12.1: Describe the key attributes that define mortgages.

A mortgage is a loan secured by real property (i.e., residential dwelling and land) as collateral and may be repossessed by the lender (mortgagor) if the borrower (mortgagee) is in default.

A servicer is compensated by the lender for collecting the payments from the borrower and forwarding them to the investors. Servicers will also liaise directly with borrowers in the event of payment delinquencies or loan foreclosure.

Lien Status

Whether the mortgage is a first lien, a second lien, or a subsequent lien will greatly impact the lender’s ability to recover the balance owing in the event of default. For example, a first lien would give the lender the first right to receive proceeds on liquidation, and so from a seniority perspective, a first lien is more desirable than a second lien.

Original Loan Term

Mortgage terms of 10 to 30 years are common, with the most popular being 30 years (long term). However, medium terms in the 10- to 20-year range are starting to become more common, given the desire of many individuals to pay off their mortgages as soon as possible.
Credit Classification

Classifying loans between prime and subprime is determined mainly by credit score (i.e., Fair Isaac Corporation or FICO model).

**Prime (A-grade) loans** constitute most of the outstanding loans. They have low rates of delinquency and default as a result of low loan-to-value (LTV) ratios (i.e., far less than 95%), borrowers with stable and sufficient income (i.e., front income ratio of no more than 28% of monthly income to service payments relating to the home and back income ratio no more than 36% for those payments plus other debt payments), and a strong history of repayments (e.g., FICO score of 660 or greater). Home payments include interest, principal, property taxes, and homeowners insurance.

*The loan-to-value ratio compares the loan amount on the property to its current fair market or appraisal value. The lower the better for this ratio from the perspective of the lender. Income ratios evaluate the level of a borrower's income level compared to the total size of the mortgage payment. Front ratios divide total monthly payments by monthly income on a pre-tax basis. Back ratios are calculated in a similar fashion, but instead add other borrower loans (such as auto loans) to the total loan payments.*

**Subprime (B-grade) loans** have higher rates of delinquency and default compared to prime loans. They could be associated with high LTV ratios (i.e., 95% or above), borrowers with lower income levels, and borrowers with marginal or poor credit histories (e.g., FICO score below 660). High LTV ratios suggest a higher risk of default. Upon issuance, subprime loans are carefully scrutinized by the servicer to ensure timely payments.

**Alternative-A loans** are the loans in between prime and subprime. Although they are essentially prime loans, certain characteristics of Alternative-A loans make them riskier than prime loans. For example, the loan value may be unusually high, the LTV ratio may be high, or there may be less documentation available (e.g., income verification, downpayment source).

**Interest Rate Type**

**Fixed-rate mortgages** have a set rate of interest for the term of the mortgage. Payments are constant for the term and consist of blended amounts of interest and principal.

**Adjustable-rate mortgages** (ARMs) have rate changes throughout the term of the mortgage. The rate is usually based on a base rate (e.g., prime rate, LIBOR) plus a spread. Rates can usually change on a monthly, semiannual, or annual basis. The risk of default is high, especially if there are large rate increases after the first year, thereby significantly increasing the total payment amount (due to the increase in interest).

There are two major types of ARMs: **fixed-period ARM** and **negative amortization ARM**.

The **fixed-period ARM** has an initial set rate that stays constant for a relatively long period of time (i.e., three to ten years) and then can change. There is an *initial rate cap*, as just described, as well as a *periodic rate cap* that sets a limit on the rate change at the end of the
initial period and future reset periods. There is also a *life rate cap* that sets a limit on the rate that can be charged for the term of the mortgage.

The **negative amortization** ARM has a very low initial set rate that adjusts on a monthly basis, but the minimum payment is adjusted only annually and there is a payment cap that limits the payment increase at reset. As a result, it is possible for the mortgage payment to be less than the interest payable when interest rates rise significantly. The loan balance is increased by the deficient interest, thereby resulting in negative amortization. There will be a significant payment increase when it is time to start amortizing the principal balance.

**Amortization Type**

Usually, all mortgages are fully amortizing in that the payments are calculated as if the mortgage will be fully paid off at the end of the amortization period. With a fixed-rate mortgage, the payments will be constant for the term of the loan. With an ARM, the payments are recalculated at each adjustment period, based on the new interest rate and the remaining principal balance. In both cases, an attempt is always made to set the payments at a level so that the principal balance will theoretically be zero at the end of the amortization period.

However, there are interest-only (IO) payment schemes now being used in the marketplace for fixed-rate mortgages. Interest-only payments are required for a set period (lockout period), and after that, the mortgage is reset to amortize the outstanding principal balance over the remaining term. The combination of adding a principal component to the payment and amortizing over a shorter period results in a dramatically higher payment amount.

The same IO payment scheme is also used for hybrid ARMs; the terms of the IO and fixed rate periods are the same. Similar to fixed-rate mortgages, the IO aspect could result in significant changes to the payment amounts due to rate increases and principal amortization.

There is also a **noncontiguous interest-only hybrid ARM**; the IO period is not the same as the fixed-rate period (assume a 10-year IO period and 5-year fixed-rate period). After the 5-year fixed-rate period is over, the loan rate is reset (assume the rate has increased). For the next five years, a higher amount of interest is paid to correspond with the new rate (and to avoid negative amortization). When the 10-year IO period is over, the mortgage payment will be further increased to account for principal amortization. The idea here is to avoid the sudden increase in payments usually associated with the simultaneous resetting and recasting of ARMs.

**Credit Guarantees**

The ability to create mortgage-backed securities requires loans that have credit guarantees.

**Government loans** are those that are backed by federal government agencies (e.g., Government National Mortgage Association or GNMA or Ginnie Mae).

**Conventional loans** could be securitized by either government-sponsored enterprises (GSEs): Freddie Mac (FHLMC) or Fannie Mae (FNMA). For a guarantee fee, these GSEs
will guarantee payment of principal and interest to the investors. Recently, cost reasons have resulted in conventional loans being securitized as private label transactions, usually through subordination. Private label transactions also arise because some loans do not meet credit or size requirements stipulated by the GSEs.

Loan Balances

The GSEs limit the loan amounts that qualify for guarantees. Any loans larger than the limit (jumbo loans) cannot be included in the GSE pools and must be securitized in private label transactions. In general, the GSE loan limits have increased over time due to real estate appreciation, so there are many more transactions being done through GSEs versus private label.

Prepayments and Prepayment Penalties

Prepayments reduce the mortgage balance and amortization period. They can occur because of the following reasons:

- Home is sold, which requires the mortgage balance to be paid off.
- Refinancing due to lower rates or more attractive loan features elsewhere.
- Partial prepayments by the borrower during the term.

To counteract the negative effects of prepayments, many loans contain prepayment penalties. They are amounts payable to the servicer for prepayments within a certain time and/or over a certain amount. Soft penalties are those that may be waived on the sale of the home; hard penalties may not be waived.

Mortgage Loan Mechanics

AIM 12.2: Calculate the mortgage payment factor.

Mortgage Payment Factor

The mortgage payment factor applies to fixed-rate mortgages because the principal balance is paid off over the term. The payment amount is constant even though the breakdown between principal and interest always changes:

$$payment_{monthly} = \frac{original\ loan\ balance \times r \times (1 + r)^T}{(1 + r)^T - 1}$$

where:
- $r = \text{monthly interest rate}$
- $T = \text{loan term (in months)}$
Example: Computing the mortgage payment factor

Assume the following information for a fixed-rate mortgage:

- Mortgage balance: $200,000
- Annual interest rate: 4.8%
- Monthly interest rate: 0.40%
- Loan term: 25 years (300 months)

Calculate the monthly payment factor and the monthly mortgage payment.

Answer:

\[
payment\ factor = \frac{0.004 \times 1.004^{300}}{(1.004)^{300} - 1} = 0.00572997
\]

Since the monthly payment factor is 0.00572997, the resulting monthly mortgage payment will be $1,146 ($200,000 \times 0.00572997).

AIM 12.3: Understand the allocation of loan principal and interest over time for various loan types.

Allocation Between Principal and Interest

Fully amortizing fixed-rate mortgage:

- The mortgage payment consists primarily of interest in the early years.
- Interest is calculated on a declining principal balance so the interest payable will gradually decrease over time. As a result, more of the fixed mortgage payment will be applied toward reducing the principal amount.
- The crossover point is the point in the mortgage where principal and interest allocation amounts are the same. After that point, relatively more amounts will be allocated to principal.
- Mortgages with shorter amortization periods result in less interest paid and more of the payment applied toward reducing the principal balance sooner. In other words, equity buildup occurs at a quicker rate when the amortization period is shorter.

Figure 1 illustrates the relationship of interest and principal over the term of the loan.
As indicated previously, the reason principal payments will increase over time is due to the reduction in the outstanding loan balance. Figure 2 illustrates the relationship between loan balance and time for a $100,000 loan.

Fixed-rate IO mortgage:

- In comparing a 5-year IO mortgage to a fully amortizing mortgage, both for a 25-year term, the principal balance outstanding after five years would be higher for the IO mortgage because none of the principal would have been paid.
- Naturally, the payments made during the first five years would be lower for an IO versus a fully amortizing mortgage.
• When the mortgage reverts to fully amortizing for the remainder of the term (240 months), the payments will be higher for the IO mortgage because the same principal amount needs to be amortized over a shorter period compared to the fully amortizing mortgage (240 months versus 300 months).

Amortizing ARM:
• The initial mortgage payment is calculated on the initial rate for the entire term. For every subsequent rate adjustment, a new monthly payment is calculated using the new rate and the remaining term.
• Assuming there is a rate increase, the mortgage payment should increase, primarily to account for the increase in interest payable. In theory, the amount allocated to the principal repayment should not really change from the amount initially calculated.

IO hybrid ARM:
• Payments are similar to fixed-rate IO mortgages.
• To illustrate, assume a $100,000 mortgage with a 5-year fixed-rate period, 10-year IO period, and 30-year term. The initial fixed rate is assumed to be 5%, which results in a monthly payment of $417 [interest only: ($100,000 × 5%) / 12]. When the loan is reset five years later at an assumed new rate of 6%, the monthly payment increases to $500 [interest only: ($100,000 × 6%) / 12]. After the IO period is over at year 10, the mortgage is recast from IO to amortizing for the remaining 20-year term. Assuming the interest rate remains at 6%, the monthly payment increases to $716 (blended interest and principal) to account for principal amortization.

Negative amortization ARM:
• Payment-option mortgages have a low initial rate for a very short period of time (one to three months). After that period, the rate changes monthly (assume it gradually increases).
• However, the minimum payment is based on the low initial rate and a 30-year amortization period, resulting in a very low payment amount. That amount does not change until after one year.
• When the mortgage is recast in one year (and every year after that), the minimum payment is increased, but the increase may not exceed 7.5%.
• As a result of interest rate increases and the 7.5% payment increase ceiling, it is quite possible that the minimum payment will be insufficient to cover the full monthly interest payments. That results in negative amortization whereby the deficient interest increases the principal balance.
• When the outstanding principal balance reaches 115% of the original mortgage amount, the mortgage is recast to amortize the outstanding balance over the remaining term. No matter what, the loan will be required to be recast after the initial 5- or 10-year period (and every five years thereafter), and payment increases will no longer be limited to the 7.5% ceiling.
Risks Associated With Mortgages and Mortgage Products

AIM 12.4: Define prepayment risk, reasons for prepayment, and the negative convexity of mortgages.

Prepayment Risk

When interest rates are falling, mortgage security holders face the risk that their relatively high-rate investment will be cashed in. Reinvestment of those funds would generally only be possible at the lower prevailing rate, thereby lowering overall investment performance.

Mortgages with a high rate of interest have higher prepayment risk because they are likely to be refinanced when interest rates fall. Of course, mortgages with a low rate of interest have lower prepayment risk because the probability of refinancing at a lower rate is small.

Additionally, the rate of prepayments will change as interest rates change, and that results in cash flow fluctuations in mortgages and mortgage securities. Hedging such investments then becomes very complex and costly.

Reasons for Prepayment

Borrowers may prepay their mortgages due to the sale or destruction of the property or a desire to refinance at lower prevailing rates. In addition, prepayments may occur because the borrower has defaulted on the mortgage and the lender is forced to sell the property to cover the mortgage. Finally, many mortgages have partial prepayment privileges (curtailment) that may be used, especially when the borrower has excess cash available to do so.

The level of refinancing activity is a function of three major factors:

- Level of interest rates.
- Yield curve shape.
- Availability of alternative financing.

Professor's Note: We will provide more detail on factors that influence prepayments in the next topic.

Negative Convexity

Regular fixed-income securities appreciate in value as interest rates decrease. With mortgage securities, however, prepayment rates increase as interest rates decrease, thereby reducing the price appreciation of mortgage securities. When interest rates increase, prepayment rates decrease, thereby increasing the relative price depreciation of mortgage securities at an amount greater than regular (non-callable) fixed-income securities.
CREDIT AND DEFAULT RISK

AIM 12.5: Explain credit and default risk analysis of mortgages, including metrics for delinquencies, defaults, and loss severity.

Credit Risk Analysis Pertaining to Mortgage Securities

Key differences from regular fixed-income securities:

- Quantifying the characteristics of the vast number of underlying loans in a mortgage security.
- Determining how those characteristics will be reported from a performance evaluation perspective; also the need to consider best-, worst-, and likely case scenarios.

Similar to regular mortgages, credit scores and LTV ratios are used to predict the credit and default risk of mortgage securities.

Delinquencies

The Office of Thrift Supervision classifies loans as follows:

<table>
<thead>
<tr>
<th>Payment due date to 29 days late</th>
<th>Current</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 to 59 days late</td>
<td>30 days delinquent</td>
</tr>
<tr>
<td>60 to 89 days late</td>
<td>60 days delinquent</td>
</tr>
<tr>
<td>90 or more days late</td>
<td>90+ days delinquent</td>
</tr>
</tbody>
</table>

Defaults

Some delinquencies may eventually lead to defaults. Default is most common for loans that are more than 90 days delinquent.

Loss Severity

Loss severity refers to the amount of loss on the loan after proceeds have been recovered from the sale of the underlying property in the foreclosure proceedings. In general, a defaulted loan with a higher LTV ratio will result in a higher loss severity. However, defaulted loans with a lower LTV ratio may have losses because the market value of the property is far less than the appraised value and/or there are significant costs, and lost income incurred in foreclosing.
1. The eight key attributes that define mortgages are lien status, original loan term, credit classification, interest rate type, amortization type, credit guarantees, loan balances, and prepayments/prepayment penalties.

2. The mortgage payment factor is calculated as:

\[
\frac{\text{interest rate} \times (1 + \text{interest rate})^{\text{loan term}}}{(1 + \text{interest rate})^{\text{loan term}} - 1}
\]

3. The allocation between principal and interest differs between fully amortizing fixed-rate mortgages, fixed-rate IO mortgages, amortizing ARMs, IO hybrid ARMs, and negative amortization mortgages. Fully amortizing mortgage payments consist of blended and level payments of interest and principal, with the interest amount decreasing and principal amount increasing over time. IO mortgages essentially result in deferred payments—lower payments in earlier years, higher in later years. ARMs carry the risk of increasing interest rates and higher interest payments.

4. Prepayment risk results when interest rates are falling and mortgage security holders face the risk that their relatively high-rate investment will be cashed in. Reasons for prepayment include sale or destruction of property, mortgage default, and curtailment. Negative convexity refers to the price compression of mortgage securities when interest rates fall due to the prepayment feature.

5. Mortgages may be current, delinquent, or in default. Some delinquencies may eventually lead to default. Default is most common for loans that are more than 90 days delinquent. Loss severity refers to the amount of loss on the loan after proceeds have been recovered from the sale of the underlying property in the foreclosure proceedings.
1. Which of the following credit classifications is most appropriate for a loan where the underlying borrower has a FICO score of 700 but is unable to provide income verification?
   A. A-grade.
   B. B-grade.
   C. C-grade.
   D. Alternative-A.

2. A fixed-rate fully amortizing mortgage was established exactly one year ago with an opening balance of $150,000 at an annual interest rate of 4.2%, with a 30-year term. Current interest rates are 4.8% per annum. Which of the following amounts represents the monthly payment (rounded) for the next year?
   A. $525.
   B. $734.
   C. $808.
   D. $860.

3. Which of the following types of mortgages is characterized by a low initial rate for a very short period of time, followed by recasting of the mortgage on an annual basis?
   A. Amortizing ARM.
   B. Negative amortization mortgage.
   C. Interest-only hybrid ARM.
   D. Noncontiguous interest-only hybrid ARM.

4. Which of the following factors would least likely affect the level of refinancing activity?
   A. Level of defaults.
   B. Yield curve shape.
   C. Level of interest rates.
   D. Availability of alternative financing.

5. Which of the following amounts generally represents the maximum number of days a loan can be past due in order for it to be considered current?
   A. 29 days.
   B. 30 days.
   C. 59 days.
   D. 60 days.
1. D Alternative-A loans are loans in between prime and subprime. Although they are essentially prime loans (i.e., FICO score 660 or greater), certain characteristics of Alternative-A loans (e.g., less documentation available and no income verification) make them riskier than prime loans.

2. B Mortgage balance: $150,900
   Annual interest rate: 4.2%
   Monthly interest rate: 0.35%
   Loan term: 30 years (360 months)

   $0.0035 \times (1.0035)^{360} \\ (1.0035)^{360} - 1
   
   = 0.004890172

   The monthly payment factor is 0.004890172, resulting in a monthly mortgage payment of $734 ($150,000 \times 0.004890172).

3. B An example of negative amortization would be a payment-option mortgage, which has a low initial rate for a very short period of time (one or three months). After that period, the rate changes monthly.

4. A The level of defaults will increase losses and reduce the return on the mortgage investment; however, the level of defaults would not have any explainable impact on the level of refinancing activity.

5. A Any loan that is late by 30 days or more is considered delinquent (and not current).
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

**Basics of Residential Mortgage-Backed Securities**

**Exam Focus**

This topic provides essentials of mortgage-backed securities (MBSs). The most crucial concept in this topic is on mortgage prepayments and how prepayments impact payments to MBS investors. For the exam, pay close attention to the cash flow structures of pass-through MBS, collateralized mortgage obligations (CMOs), and interest-only and principal-only strips.

**Securitization**

AIM 13.1: Summarize the securitization process of residential mortgage backed securities (MBS).

To reduce the risk from holding a potentially undiversified portfolio of mortgage loans, a number of financial institutions (i.e., originators) will work together to pool residential mortgage loans with similar characteristics into a more diversified portfolio. They will then sell the loans to a separate entity, called a special purpose vehicle (SPV), in exchange for cash. An issuer will purchase those mortgage assets in the SPV and then use the SPV to issue mortgage-backed securities (MBSs) to investors; the securities are backed by the mortgage loans as collateral.

As of now, the securitization process has become a crucial part of the U.S. credit system. Financial institutions expect to originate mortgage loans and sell them through securitization. The lack of a securitization market for mortgages would lead to the downfall of mortgage lending because financial institutions would not want to retain the risks.

**Agency and Non-Agency MBS**

AIM 13.2: Differentiate between agency and non-agency MBS and describe the major participants in the residential MBS market.

**Agency MBS**

Agency MBSs are those that are guaranteed by any of three government-sponsored entities (GSEs): Ginnie Mae (GNMA), Fannie Mae (FNMA), or Freddie Mac (FHLMC). Most of the MBSs are issued by these GSEs.
Ginnie Mae’s primary role is to guarantee the timely payments of MBSs backed by loans made through the Federal Housing Administration program, the Office of Public and Indian Housing program, and the Department of Veteran Affairs Home Loan program. The guarantee makes the MBS absolutely default free.

Fannie Mae is a large issuer of MBSs. It has a large portfolio of mortgage loans and issues debt to finance the portfolio. One of the benefits of issuing debt is to add liquidity to the mortgage market and, therefore, allow financial institutions to lend at lower rates. Fannie Mae also provides credit guarantees on mortgages that it securitizes. MBSs issued or guaranteed by Fannie Mae are reasonably default free.

Freddie Mac is similar to Fannie Mae from an operational perspective. Its objective is to stabilize U.S. residential markets and expand opportunities for home ownership and affordable rental housing. As with Fannie Mae, MBSs issued or guaranteed by Freddie Mac are reasonably default free.

Non-Agency MBS

Also known as private label, the non-agency MBS segment grew along with U.S. home prices over time up to the 2007 credit crisis. The GSEs have restrictions on what mortgages they can guarantee/securitize [e.g., dollar value limit, loan-to-value (LTV) ratio limit], which opened up the private label market for those participants willing to take on the risks inherent in nonconventional loans—jumbo loans (mortgage principal balance over the limit) and/or loans with high LTVs. The rising prices of the underlying homes held as collateral provided some risk mitigation.

Unfortunately, the falling prices of homes and the credit crisis beginning in 2007 caused a significant drop in MBS issuances in the non-agency segment because they did not have government guarantees.

Default Risk

With agency MBSs, the investor bears no credit risk because the GSEs have been paid a fee to guarantee the underlying mortgages. If there is a default with a mortgage, the GSE will pay the outstanding balance to the investors. With a non-agency MBS, there is some credit risk, but that is mitigated through the process of subordination (to be discussed in greater detail in the next topic).

Prepayment

AIM 13.3: Describe the mortgage prepayment option and the factors that influence prepayments.

Mortgage Prepayment Option

Mortgage prepayments come in two forms: (1) increasing the frequency or amount of payments (where permitted) and (2) repaying/refinancing the entire outstanding balance.
Prepayments are much more likely to occur when market interest rates fall and borrowers wish to refinance their existing mortgages at a new and lower rate. For the lender, prepayments represent a loss for two reasons: (1) they stop receiving interest income at the high rate and (2) they have to reinvest the proceeds received from prepayment at the prevailing lower market rates. Therefore, the pricing of the initial mortgage rate should be somewhat higher to take into account the possibility of prepayment. With agency MBSs, prepayments and defaults have the same impact on investors. Prepayments result in the investors actually receiving cash from the borrowers, whereas with defaults, the borrower does not pay the outstanding mortgage balances, but the GSE does, thereby causing a prepayment.

Other Factors That Influence Prepayments

Seasonality. The summertime is a popular time for individuals to move (and mortgages must be paid out prior to the sale of a home), so it is the period of time with the greatest prepayment risk. Given some time lags, the prepayments often start to appear in the late summer and early fall.

Age of mortgage pool. Refinancing often involves penalties and administrative charges, so borrowers tend not to do so until several years into the mortgage. Also, it takes some time for borrowers to build up equity and savings to make prepayments and/or attempt to refinance. As a result, the lower the age of the mortgage pool, the less likely the risk of prepayment.

Personal. Marital breakdown, loss of employment, family emergencies, and destruction of property are commonly cited reasons for prepayments based on personal reasons. It is difficult to assess this type of prepayment risk.

Housing prices. Property value increases may spur an increase in prepayments caused by borrowers wanting to take out some of the increased equity for personal use. Property value decreases reduce the value of collateral, reduce the ability to refinance, and, therefore, decrease the risk of prepayment. The increasingly popular use of home equity lines of credit where the mortgage balance is revolving (i.e., mortgage balance can be drawn up to a certain limit and paid down to zero at any time) reduces refinancing and prepayment risk due to the nature of the loan.

Refinancing burnout. To the extent that there has been a significant amount of prepayment or refinancing activity in the mortgage pool in the past, the risk of prepayment in the future decreases. That is because presumably the only borrowers remaining in the pool are those who were unable to refinance earlier (e.g., due to poor credit history or insufficient property value), and those who did refinance have been removed from the pool already. Also, those who made only large prepayments (instead of fully refinancing) in the past would have exhausted their savings to make the prepayment and would require quite some time to do so again in the future.
AIM 13.4: Describe the impact on a MBS of the weighted average maturity, the weighted average coupon, and the speed of prepayments of the mortgages underlying the MBS.

AIM 13.5: Identify, describe, and contrast different standard prepayment measures.

Mortgage Pass-Through Securities

A mortgage pass-through security represents a claim against a pool of mortgages. Any number of mortgages may be used to form the pool, and any mortgage included in the pool is referred to as a securitized mortgage. The mortgages in the pool have different maturities and different mortgage rates. The weighted average maturity (WAM) of the pool is equal to the weighted average of all the mortgages in the pool, each weighted by the relative outstanding mortgage balance to the value of the entire pool. The weighted average coupon (WAC) of the pool is the weighted average of the mortgage rates in the pool. The investment characteristics of a mortgage pass-through are a function of its cash flow features and the strength of its government guarantee.

As illustrated in Figure 1, pass-through security investors receive the monthly cash flows generated by the underlying pool of mortgages, less any servicing and guarantee/insurance fees. The fees account for the fact that pass-through rates (i.e., the coupon rate on the pass-through) are less than the average coupon rate of the underlying mortgages in the pool.

**Figure 1: Mortgage Pass-through Cash Flow**

Since pass-through securities may be traded in the secondary market, they effectively convert illiquid mortgages into liquid securities (as mentioned, this process is called securitization). As we will see later in this topic, more than one class of pass-through securities may be issued against a single mortgage pool.

The timing of the cash flows to pass-through security holders does not exactly coincide with the cash flows generated by the pool. This is due to the delay between the time the mortgage service provider receives the mortgage payments and the time the cash flows are “passed through” to the security holders.

The most important characteristic of pass-through securities is their prepayment risk; because the mortgages used as collateral for the pass-through can be prepaid, the pass-throughs themselves have significant prepayment risk.
Measuring Prepayment Speeds

Prepayments cause the timing and amount of cash flows from mortgage loans and MBS to be uncertain; they speed up principal repayments and reduce the amount of interest paid over the life of the mortgage. Thus, it is necessary to make specific assumptions about the rate at which prepayment of the pooled mortgages occurs when valuing pass-through securities. Two industry conventions have been adopted as benchmarks for prepayment rates: the conditional prepayment rate (CPR) and the Public Securities Association (PSA) prepayment benchmark.

The CPR is the annual rate at which a mortgage pool balance is assumed to be prepaid during the life of the pool. A mortgage pool’s CPR is a function of past prepayment rates and expected future economic conditions.

We can convert the CPR into a monthly prepayment rate called the single monthly mortality rate (SMM) (also referred to as constant maturity mortality) using the following formula:

\[ SMM = 1 - (1 - CPR)^{1/12} \]

An SMM of 10% implies that 10% of a pool’s beginning-of-month outstanding balance, less scheduled payments, will be prepaid during the month.

The PSA prepayment benchmark assumes that the monthly prepayment rate for a mortgage pool increases as it ages, or becomes seasoned. The PSA benchmark is expressed as a monthly series of CPRs.

The PSA standard benchmark is referred to as 100% PSA (or just 100 PSA). 100 PSA (see Figure 2) assumes the following graduated CPRs for 30-year mortgages:

- CPR = 0.2% for the first month after origination, increasing by 0.2% per month up to 30 months. For example, the CPR in month 14 is 14 (0.2%) = 2.8%.
- CPR = 6% for months 30 to 360.

Figure 2: 100 PSA

![Diagram of 100 PSA](image)
Remember that the CPRs are expressed as annual rates.

A particular pool of mortgages may exhibit prepayment rates faster or slower than 100% PSA, depending on the current level of interest rates and the coupon rate of the issue. A 50% PSA refers to one-half of the CPR prescribed by 100% PSA, and 200% PSA refers to two times the CPR called for by 100% PSA.

**Example: Computing the SMM**

Compute the CPR and SMM for the 5th and 25th months, assuming 100 PSA and 150 PSA.

**Answer:**

Assuming 100 PSA:

\[
\begin{align*}
\text{CPR (month 5)} &= 5 \times 0.2\% = 1\% \\
100 \text{ PSA} &= 1 \times 0.01 = 0.01 \\
\text{SMM} &= 1 - (1 - 0.01)^{1/12} = 0.000837
\end{align*}
\]

\[
\begin{align*}
\text{CPR (month 25)} &= 25 \times 0.2\% = 5\% \\
100 \text{ PSA} &= 1 \times 0.05 = 0.05 \\
\text{SMM} &= 1 - (1 - 0.05)^{1/12} = 0.004265
\end{align*}
\]

Assuming 150 PSA:

\[
\begin{align*}
\text{CPR (month 5)} &= 5 \times 0.2\% = 1\% \\
150 \text{ PSA} &= 1.5 \times 0.01 = 0.015 \\
\text{SMM} &= 1 - (1 - 0.015)^{1/12} = 0.001259
\end{align*}
\]

\[
\begin{align*}
\text{CPR (month 25)} &= 25 \times 0.2\% = 5\% \\
150 \text{ PSA} &= 1.5 \times 0.05 = 0.075 \\
\text{SMM} &= 1 - (1 - 0.075)^{1/12} = 0.006476
\end{align*}
\]
Figure 3: Prepayment Speeds for 5th and 25th Months at 100 and 150 PSA

It is important for you to recognize that the nonlinear relationship between CPR and SMM implies that the SMM for 150% PSA does not equal 1.5 times the SMM for 100% PSA. Also, keep in mind that the PSA standard benchmark is nothing more than a market convention. It is not a model for predicting prepayment rates for MBS. In fact, empirical studies have shown that actual CPRs differ substantially from those assumed by the PSA benchmark.

**Duration and Convexity**

AIM 13.6: Describe the effective duration and effective convexity of standard MBS instruments and the factors that affect them.

**Effective Duration**

The calculation of effective duration for standard pass-through MBS is different than for regular fixed-income securities due to the increased refinancing activity usually associated with a decrease in interest rates. In such cases, the sensitivity of the price to a change in interest rate will be lower than a regular fixed-income security due to the prepayments. It is important from a duration-based hedging perspective that the duration estimate be calculated properly or else the hedge will be ineffective and potentially result in losses.
The effective duration approximation is given as follows:

\[
\frac{1}{P} \times \frac{P_{+\Delta y} - P_{-\Delta y}}{2 \times \Delta y}
\]

where:

- \( P \) = current price of MBS
- \( P_{+\Delta y} \) = price of MBS for a given increase in interest rates (parallel shift)
- \( P_{-\Delta y} \) = price of MBS for a given decrease in interest rates (parallel shift)

Note that the prices of the MBS must consider the change in prepayment speed that would occur with the change in interest rates.

**Effective Convexity**

Since a decrease in interest rates will increase the rate of prepayments, the price of the MBS will not increase as much correspondingly. That price compression results in negative convexity (similar to that of a callable bond), which is a risk for investors in the form of higher average negative capital gains (i.e., positive capital losses). Partly as a result of the increased risk, MBS will provide a higher yield compared to Treasury securities. In other words, think of the additional yield as an option premium received by the investors for writing a call option or as compensation for the expected negative capital gain.

The convexity approximation is given as follows:

\[
\frac{1}{P} \times \frac{P_{+\Delta y} - P_{-\Delta y} - 2 \times P}{\Delta y^2}
\]

where:

- \( P \) = current price of MBS
- \( P_{+\Delta y} \) = price of MBS for a given increase in interest rates (parallel shift)
- \( P_{-\Delta y} \) = price of MBS for a given decrease in interest rates (parallel shift)

**Collateralized Mortgage Obligations (CMOs)**

AIM 13.7: Describe collateralized mortgage obligations (CMOs) and contrast them with MBSs.

CMOs are structured in a way so as to offer varying levels of prepayment risk, depending on the investors' risk and return preferences. As a result, it may increase the liquidity of such securities compared to MBSs. CMOs are more complex than the standard MBS pass-through. In contrast to a pass-through structure, a CMO allows for investors to alter their exposure to prepayment risk. In addition, CMOs pay quarterly cash flows compared to MBSs, which generally pay monthly cash flows.

Agency CMOs are guaranteed, but non-agency CMOs are not, so non-agency CMOs require overcollateralization (i.e., amount of assets in the mortgage pool exceeds the amount of securities issued) in order to be more marketable.
MBS Cash Flow Examples

AIM 13.8: Describe and work through a simple cash flow example for the following types of MBS:
- Pass-through securities.
- CMOs, both sequential and planned amortization class.
- Interest only and principal only strips.

Pass-Through Securities

The following is a numerical example to illustrate and explain the first two months of cash flows on a pass-through security, assuming:

- MBS principal balance $600 million
- Original mortgage pool WAM 360 months
- Original mortgage pool WAC 6.5%
- Annualized pass-through coupon 6.0%
- Prepayment assumption 200% PSA

Figure 4: Cash Flow Analysis of a Pass-Through Security

<table>
<thead>
<tr>
<th>Month</th>
<th>CPR  (Note 1)</th>
<th>SMM  (Note 2)</th>
<th>Coupon  (Note 3)</th>
<th>Mortgage Interest (Note 4)</th>
<th>Scheduled Principal (Note 5)</th>
<th>Prepaid Principal (Note 6)</th>
<th>Pass-Through Interest (Note 7)</th>
<th>Total Cash Flow (Note 8)</th>
<th>Ending Principal (Note 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.20%</td>
<td>0.03%</td>
<td>3.79</td>
<td>3.25</td>
<td>0.54</td>
<td>0.20</td>
<td>3.00</td>
<td>3.74</td>
<td>599.26</td>
</tr>
<tr>
<td>2</td>
<td>0.40%</td>
<td>0.07%</td>
<td>3.79</td>
<td>3.25</td>
<td>0.54</td>
<td>0.40</td>
<td>3.00</td>
<td>3.94</td>
<td>598.32</td>
</tr>
</tbody>
</table>

The information given in Figure 1 is determined as follows:

1. Standard assumption per PSA: 0.20% prepaid in first month and increases by 0.20% in the second month.

2. Month 1: SMM = 1 − (1 − CPR)\(^{1/12}\) = 1 − (1 − 0.002)\(^{1/12}\) = 0.0167%. At 200% PSA, SMM = 0.0167% × 2 = 0.0334%.
   Month 2: SMM = 1 − (1 − CPR)\(^{1/12}\) = 1 − (1 − 0.004)\(^{1/12}\) = 0.0334%. At 200% PSA, SMM = 0.0334% × 2 = 0.0668%.

3. Month 1: PV = 600, i = 6.5% / 12, n = 360. Solve for PMT (blended principal and interest) = 3.79. Taking into account prepayments in month 1, the updated coupon for month 2: (1 − 0.0003) × 3.79 = 3.7889.

4. Month 1: WAC / 12 × outstanding principal = 0.065 / 12 × 600 = 3.25.  
   Month 2: WAC / 12 × outstanding principal = 0.065 / 12 × 599.26 = 3.246.

5. Coupon – interest. Month 1: 3.79 – 3.25 = 0.54. Month 2: 3.7889 – 3.246 = 0.543.

6. Month 1: Pass-through coupon / 12 × outstanding balance = 0.06 / 12 × 600 = 3.00.  
   Month 2: Pass-through coupon / 12 × outstanding balance = 0.06 / 12 × 599.26 = 2.996.
7. Month 1: Scheduled principal + prepaid principal + pass-through interest = 0.54 + 0.20 + 3.00 = 3.74.  
Month 2: Scheduled principal + prepaid principal + pass-through interest = 0.543 + 0.40 + 2.996 = 3.939.

8. Month 1: Opening principal – scheduled principal – prepaid principal = 600 – 0.54 – 0.20 = 599.26.  
Month 2: Ending principal (month 1) – scheduled principal – prepaid principal = 599.26 – 0.543 – 0.40 = 598.32.

Sequential CMOs

For a sequential-pay CMO, principal repayments are divided into numerous tranches. Each tranche receives interest cash flows and possibly principal cash flows. Receipt of principal cash flows (from both scheduled payments and unscheduled prepayments from the underlying mortgages) depends on the stage of principal repayments within the tranches—the first tranche's principal has to be fully repaid before the next tranche receives any principal, and so on.

Occasionally, a Tranche Z is in place (also referred to as the equity tranche). It receives no cash flows, but the interest it would have otherwise received periodically is added to the principal to be paid later. When all the other tranches have been repaid their principal, Tranche Z investors will begin to receive principal.

The following continues the numerical example from pass-through securities with additional assumptions to illustrate the sequential aspect:

- Total cash flows are subdivided into four tranches: A, B, C, and D.
- Total principal of $600 million is also subdivided: Tranche A has $250 million principal, B has $150 million, C has $125 million, and D has $75 million.
- Annualized pass-through coupon remains at 6%.

In month 1, Tranche A investors receive interest of ($250 million × 6%) / 12 = $1.25 million. They also receive all scheduled principal payments and prepayments of $0.54 million and $0.20 million. Total cash flow is $1.99 million.

Tranches B, C, and D receive interest only and no principal until all of Tranche A's principal has been paid out. In month 1, Tranche B investors receive interest of ($150 million × 6%) / 12 = $0.75 million. Tranche C investors receive interest of ($125 million × 6%) / 12 = $0.625 million. Tranche D investors receive interest of ($75 million × 6%) / 12 = $0.375 million.

To illustrate the sequential nature, assume Tranche A's principal is fully paid out in month 60. Then Tranche B's principal starts to be repaid in month 60. Assume Tranche B is fully paid out in month 105. At that time, Tranche C's principal starts to be repaid in month 105. Assume Tranche C is fully paid out in month 180. Finally, Tranche D's principal starts to be repaid in month 180 until it is fully paid out.
Based on this information, it is important to note the following:

- The allocation of principal repayments has a significant impact on effective duration. For example, the duration for Tranche A is significantly lower than that of Tranche D because more of the principal is being repaid sooner in Tranche A.
- The duration of the entire pass-through is simply the weighted average of the individual tranches' durations.
- Changes in prepayment speeds will impact the timing of the retirements and effective durations of each tranche. For example, if the prepayment speed assumption is decreased, then the individual tranches will be retired later, and they will have larger effective durations.

**CMO Planned Amortization Class (PAC)**

In this type of CMO structure, there is at least one PAC tranche and one support tranche. The payments to the PAC tranche(s) are held constant, and so the support tranche exists to absorb actual prepayments that are higher or lower than planned. There is generally no prepayment risk for the PAC tranches and a significant amount of prepayment risk for the support tranche (an exception will be discussed later where the support tranche has been retired due to excessive prepayments).

The following continues the numerical example from pass-through securities with additional assumptions to illustrate the PAC aspect:

- Total cash flows are subdivided into two tranches: Tranche A (or the PAC tranche) and support tranche.
- PSA range is assumed to be between 80% and 300%, and without getting into the calculations, it results in principal amounts of $356.69 million and $243.31 million in the PAC tranche and support tranche, respectively.

The scheduled cash flow for the PAC tranche will be set at the minimum between the cash flows under the 80% PSA and 300% PSA assumptions. This will ensure that changes in prepayment speed of the original pool generate cash flow amounts that can always be absorbed by the support tranche.

**Scenario 1: Actual (true) PSA is 100%.

- Actual cash flows from the pass-through less the amounts scheduled to PAC tranche holders are absorbed by the support tranche.**

**Scenario 2: Actual (true) PSA is 200%.

- Cash flows from the pass-through are much larger than in Scenario 1.
- The amounts scheduled to PAC tranche holders will stay the same as in Scenario 1 (this is the protection that is afforded as a result of having a support tranche).
- The excess amounts received will be absorbed by the support tranche.
Scenario 3: Actual (true) PSA is 300%.

- Same as in Scenario 2 but even larger cash flows.
- However, there is a limit as to how much can be absorbed by the support tranche. Once the cumulative amount of prepayments exceeds the original principal amount of $243.31 million, the support tranche will be retired, and the PAC tranche’s cash flows will no longer be protected.
- PAC tranche cash flows will now be the same as a regular pass-through security and subject to the same prepayment risks.

Scenario 4: Actual (true) PSA is more than 300% (above the assumed range).

- Similar to Scenario 3 but the support tranche will be repaid and retired even sooner, and the PAC tranche will be subject to prepayment risk even sooner.

From a valuation perspective, as long as the PSA stays between 80% and 300% and the support tranche still exists, there will be no change in value to the PAC tranche due to PSA changes because the cash flows to the PAC tranche remain at the scheduled amount. However, there will be significant changes to the price of the support tranche due to the significant cash flow changes. In fact, the price change will exhibit negative convexity due to the high negative correlation between changes in interest rates (decrease) and changes in PSA (increase).

**Interest-Only (IO) Strips**

With IO strips, investors receive only the interest cash flows from the underlying pool of mortgages and none of the principal. Consequently, falling interest rates will increase the rate of prepayments and, therefore, the amount of future interest cash flows due to the reduced amount of principal on which interest is calculated. In fact, the value of IOs falls dramatically when interest rates fall and PSA rises. The effective duration of IO strips is significant and negative. That also implies a strong positive correlation between interest rates and the value of IO strips.

**Principal-Only (PO) Strips**

PO strips are the exact opposite of IO strips, whereby investors only receive the principal (scheduled and unscheduled) cash flows and none of the interest. Again, falling interest rates will increase the rate of prepayments and also the short-term cash flow. The effective duration of PO strips is significant and positive. That also implies a strong negative correlation between interest rates and the value of PO strips.

The price/yield relationships for IO and PO securities are shown in Figure 5. Notice the following:

- The underlying pass-through security exhibits significant negative convexity.
- The PO exhibits some negative convexity at low rates.
- The IO price is positively related to mortgage rates at low current rates.
- The PO and IO prices are more volatile than the underlying pass-through.
Figure 5: Investment Characteristics of IOs and POs

Price ($)
**Key Concepts**

1. To reduce the risk from holding a potentially undiversified portfolio of mortgage loans, a number of financial institutions (originators) will work together to pool residential mortgage loans with similar characteristics into a more diversified portfolio. They will then sell the loans to a separate entity, called a special purpose vehicle (SPV), in exchange for cash. An issuer will purchase those mortgage assets in the SPV and then use the SPV to issue mortgage-backed securities (MBSs) to investors; the securities are backed by the mortgage loans as collateral.

2. Agency MBSs are those that are guaranteed by any of three GSEs: Ginnie Mae, Fannie Mae, or Freddie Mac. Most of the MBSs are issued by these GSEs.

   The GSEs have restrictions on which mortgages they can guarantee/securitize, which opened up the private label market for those participants willing to take on the risks inherent in nonconventional loans—jumbo loans and/or loans with high loan-to-value ratios.

3. Mortgage prepayments come in two forms: (1) increasing the frequency or amount of payments and (2) repaying/refinancing the entire outstanding balance. Prepayments are much more likely to occur when market interest rates fall and borrowers wish to refinance their existing mortgages at a new and lower rate.

   Other factors that influence prepayments include seasonality, age of mortgage pool, personal, housing prices, and refinancing burnout.

4. The value of an MBS is a function of:
   - Weighted average maturity.
   - Weighted average coupon.
   - Speed of prepayments.

5. Regarding prepayment speeds, the single monthly mortality rate (SMM) is derived from the conditional prepayment rate and is used to estimate monthly prepayments for a mortgage pool:

   \[ SMM = 1 - (1 - CPR)^{1/12} \]

6. The approximation of effective duration is given as follows:

   \[ \frac{1}{P} \times \frac{P_{+\Delta y} - P_{-\Delta y}}{2 \times \Delta y} \]

   The approximation of effective convexity is given as follows:

   \[ \frac{1}{P} \times \frac{P_{+\Delta y} - P_{-\Delta y} - 2 \times P}{\Delta y^2} \]

7. CMOs are structured in a way so as to offer varying levels of prepayment risk, depending on the investors' risk and return preferences. As a result, it may increase the liquidity of such securities compared to MBSs. CMOs are more complex than the standard MBS pass-through.
8. Regarding cash flow structures for MBSs:
   • Cash flows are reasonably straightforward for pass-through securities, with all investors equally exposed to prepayment risk.
   • CMOs, both sequential and planned amortization class (PAC), involve setting up various tranches that result in prepayment risk being shifted between the investors. Sequential tranches receive principal payments sequentially over time. PAC tranches receive principal payments according to a specific formula, and any excesses/shortages are absorbed by a support tranche.
   • Interest-only and principal-only strips demonstrate significant negative duration and significant positive duration, respectively.
1. Which of the following statements about agency and non-agency MBS is correct?
   A. MBSs issued and guaranteed by Ginnie Mae are absolutely default free.
   B. MBSs issued or guaranteed by Freddie Mac are absolutely default free.
   C. Operationally, Fannie Mae and Freddie Mac follow a similar business model.
   D. With a non-agency MBS, credit risk is usually mitigated by paying a third party to guarantee the underlying loans.

2. Which of the following factors is least likely to influence the level of residential mortgage prepayments?
   A. Seasonality.
   B. Inflation.
   C. Housing prices.
   D. Age of mortgage pool.

3. If the conditional prepayment rate (CPR) for a pool of mortgages is assumed to be 5% on an annual basis and the weighted average maturity of the underlying mortgages is 15 years, which of the following amounts is closest to the constant maturity mortality?
   A. 0.333%.
   B. 0.405%.
   C. 0.427%.
   D. 0.5%.

4. Which of the following statements regarding standard MBS pass-throughs and CMOs is correct?
   A. CMOs pay quarterly cash flows.
   B. CMOs are generally less liquid securities than standard MBS pass-throughs.
   C. Agency standard MBS pass-throughs often make use of the overcollateralization feature to increase marketability.
   D. Agency and non-agency standard MBS pass-throughs allow for investors to alter their exposure to prepayment risk.

5. Which of the following statements describes the price compression of MBSs that results in negative convexity?
   A. An increase in interest rates increases the rate of residential mortgage prepayments.
   B. An increase in interest rates decreases the rate of residential mortgage prepayments.
   C. A decrease in interest rates increases the rate of residential mortgage prepayments.
   D. A decrease in interest rates decreases the rate of residential mortgage prepayments.
CONCEPT CHECKER ANSWERS

1. C  Ginnie Mae does not issue MBSs; it only guarantees them. MBSs issued or guaranteed by Freddie Mac are only relatively default free (unlike those guaranteed by Ginnie Mae, which are technically absolutely default free). With a non-agency MBS, credit risk is usually mitigated through the process of subordination.

2. B  Seasonality does impact the level of prepayments—they are noticeably higher in the summertime. Increases in housing prices may spur an increase in prepayments caused by refinancing mortgages stemming from borrowers wanting to take out some of the increased equity for personal use. The lower the age of the mortgage pool, the less likely the risk of prepayment.

3. C  The constant maturity mortality (or single monthly mortality rate) is a monthly measure. Its relationship to CPR is as follows:

   \[ SMM = 1 - (1 - CPR)^{1/12} = 1 - (1 - 0.05)^{1/12} = 1 - 0.95^{1/12} = 0.43\% \]

4. A  Because CMOs are structured in a way so as to offer varying levels of prepayment risk to investors, the liquidity of CMOs may be higher than that of standard MBS pass-throughs. Agency MBSs are guaranteed, so there is no need to overcollateralize them; overcollateralization would occur with non-agency MBSs. Agency and non-agency MBSs do not allow investors to alter their prepayment risk exposure; it is only available with CMOs. CMOs pay quarterly cash flows compared to MBSs, which generally pay monthly cash flows.

5. C  Because a decrease in interest rates will increase the rate of prepayments, the price of the MBS will not increase as much correspondingly. That price compression results in negative convexity (similar to that of a callable bond), which is a risk for investors in the form of higher average negative capital gains (i.e., positive capital losses).
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP®. This topic is also covered in:

OVERVIEW OF THE MORTGAGE-BACKED SECURITIES MARKET

Topic 14

EXAM FOCUS

This topic builds on the terminology and material you studied in the previous topic, so you need to ensure you have a solid understanding of those basics before moving on. For this topic, we will go into more detail regarding agency and private label MBS pools, pass-throughs, CMOs, and mortgage strips.

Evolution of the MBS Market

AIM 14.1: Describe the evolution of the MBS market.

Strong growth in the real estate market has led to the corresponding growth of the mortgage-backed securities (MBSs) market, as shown by the wide range and diversity of MBS products currently in existence. The MBS market has now exceeded the size of the Treasury market. Investment managers are carefully studying MBS markets and realizing the importance of analyzing the factors that impact MBS returns.

Mortgage loans are increasingly being securitized to access the capital markets for funding and liquidity. The general process works as follows: mortgage lenders approve loan applications, issue loans to borrowers, and then sell the loans in the capital markets in the form of MBSs. Proceeds from the sales of MBSs are used for new mortgage lending.

Many mortgage loans with similar characteristics (e.g., interest rate, term, credit quality, balance) are pooled together to create MBSs that can be traded. The process essentially transformed what was once a small and relatively illiquid asset into a large and liquid one.

Prior to the existence of the MBS market, financial institutions accepted deposits and then loaned them out. Those loans remained as investment assets on their books. Unfortunately, profitability was highly dependent on local economic conditions, and a shortage of local deposit funds could effectively eliminate the opportunity to issue mortgage loans. However, since the early 1980s, mortgage lending has grown significantly into a national industry where there is an active market for mortgage securities from its origins as a localized industry.


AGENCY AND PRIVATE-LABEL SECURITIZATION

AIM 14.2: Explain the creation of agency (fixed rate and adjustable rate) and private-label MBS pools, pass-throughs, CMOs, and mortgage strips.

MBSs can be created in two ways. First, the loans that meet the government agency requirements are charged an insurance premium (guarantee fee) by the agency and then securitized as a pool. Alternatively, the loans that do not meet the requirements or where it is too costly to go through an agency are securitized in non-agency or private label transactions. Although there is no agency guarantee on those securities per se, there is insurance in the form of a private guarantee or the creation of subordinate classes (the senior classes are given the greatest protection).

After a pool of mortgages is securitized, it is sold to investors as a pass-through investment. A basic example would have the principal and interest paid based on proportional ownership of the pool. A more common but complex example would involve the creation of structured securities where the pool’s cash flows are divided up (into tranches) to create many different types of securities to meet the varying needs of investors. Those securities could have different durations, prepayment risks, and credit risks.

Agency Fixed-Rate Pools and Pass-Throughs

The lender will go through the large portfolio of funded mortgages and group the ones with the same coupon to form pools. The lender transfers a certain amount of loans with a certain coupon to the agency and in return will receive the same amount back in MBSs. The MBSs received could either consist solely of a pool of the lender’s own loans or contain a portion of a pool of loans from other lenders as well.

Agency fixed-rate MBSs are traded based on coupon rates, normally in 50 bp increments (although 25 and 12.5 bp increments are possible). Some of the interest cash flows are required for servicing and credit support payments.

- **Guarantee fees** are paid to government agencies to insure the loans. They will vary based on the assessed risk of the loans. High-volume transactions tend to produce lower fees.
- **Required (base) servicing fees** are paid to servicers to perform tasks such as collecting payments, making tax and insurance payments, and forwarding payments to investors. Fees are approximately 25 bp.
- **Excess servicing** (remaining interest) is equal to:

  \[ \text{loan (note) rate} - \text{servicing fees} - \text{guarantee fees} - \text{pass-through pool coupon rate} \]

In general, the loan (note) rate must be higher than the pass-through pool coupon rate. That excess is normally in the 25 to 75 bp range. Anything higher would be suboptimal in that excess servicing would be retained. It is also possible to simply pay the GSE the guarantee fee as a capitalized upfront amount so that on an on-going basis, the pass-through pool coupon rate could be made higher to attract more investors.
Agency Adjustable-Rate Pools and Pass-Throughs

The lender's loans in the pool may have a range of rates, so the pool's coupon will be a weighted average of the loan (note) rates and the loan balances [i.e., the weighted average coupon (WAC)]. As a result, guarantee fees are usually not paid upfront but are simply paid over time. Pools may contain loan (note) rates lower than the pool coupon rate, in which case there is no need for excess servicing. Base servicing fees are in the range of 12.5 to 37.5 bp.

Adjustable-rate pools are traded in odd coupons up to three decimal places. In addition, the coupons will change over time as loans in the pool are paid off.

Private Label Pools and Pass-Throughs

In private label pools and pass-throughs, no guarantee fee is involved so there needs to be some method of credit enhancement. The most common method is the use of subordination where a portion of the transaction is subordinate in terms of receiving the cash flows. That portion will be the first to absorb losses to protect the senior portion.

Within the subordinated portion, it may be divided into separate tranches, each with separate ratings and exposure to credit losses. Compared to senior tranches, subordinate tranches carry more risk and likelihood of credit losses, so the subordinate tranches require a higher rate of return. Given the protection provided to them, the senior tranches have very predictable cash flows, just like agency pass-throughs.

The rating agencies determine the level of subordination and relative sizes of the subordinated tranches required for the deal. They are decided based on a determination of the probability and severity of losses.

Shifting interest structures are common with subordinated deals. General features are as follows:

- Prepayments are initially given to the senior tranches.
- Subordinate tranches do not initially receive prepayments but do receive the scheduled principal payments.
- Over time, the subordinate tranches increase in size, further protecting the senior tranches.
- Eventually, the subordinate tranches will receive pro rata shares of prepayments.

Overcollateralization is used when default risk is higher (i.e., for subprime loans). It is structured so that the amount of loan collateral exceeds the investment amount. Again, this provides more protection to the senior tranches.

Agency Mortgage Strips

There are principal-only (PO) and interest-only (IO) securities that are created by placing agency pools into a trust and breaking up the two components of the mortgage cash flows. PO security holders receive principal prepayments in addition to the regular principal payments. IO security holders receive the interest earned from the mortgages.
POs perform well when interest rates fall because there will be more cash inflows as a result of mortgage prepayments. IOs perform well when rates rise, and prepayments fall because there will be continued generation of interest income due to the lack of prepayments.

Private Label Mortgage Strips

WAC IOs and WAC POs exist only in the private label market. The actual coupon rate for these securities depends on market conditions (i.e., investors’ required return and prepayment outlook). Once the deal’s coupon rate is set, the underlying loans are classified into discount and premium loan groups. The base servicing and trustee fees are deducted from each note’s rate to arrive at a net note rate. Loans with a net note rate less than the set coupon rate are discount loans; those with a net note rate greater than the set coupon rate are premium loans.

The two loan groups then need to be altered to match the deal’s coupon rate. The discount loans are grossed up to the deal’s coupon rate by creating an amount of PO as follows:

\[
PO \text{ percentage} = \frac{(\text{deal coupon} - \text{note rate})}{\text{deal coupon}}
\]

The PO percentage for each note rate is multiplied by its face value. The total of all POs created for all the rates will become the size of the WAC PO.

On the other hand, the premium loans have some of their interest removed so as to reduce their net note rates to that of the deal coupon rate. To illustrate, consider $50 million face value of loans with a 5% note rate with the deal coupon rate of 4.5%. Total fees are 30bp, so that gives a net note rate of 4.7%. As a result, 20bp is stripped from the loans, thereby creating a strip with a notional value of $50 million and 0.2% coupon. The notional value of the WAC IO is the total notional value of the premium loans. The coupon of the WAC IO is the weighted average of the strip coupons with their notional values.

Based on this information, if the deal’s coupon rate is changed, it will alter the allocation of loans classified as discount or premium. Therefore, it will change the size of the WAC IO and WAC PO as well as the WAC IO’s coupon rate. The deal coupon rate is dependent on:

- Investor demand for premium and discount coupons.
- Investor demand for IOs and POs.

Agency CMOs

Agency CMOs are established with the underwriter purchasing agency MBS pools who essentially places them in a trust. Different tranches are created from the principal and interest from the MBS pools.

Private Label CMOs

Private label CMOs are established by transferring many loans into a securitization vehicle. The structured transaction (CMO) is then created by the issuer.
CREATING AN MBS

AIM 14.3: Understand how a loan progresses from application to agency pooling.

There are four steps involved in the MBS progression, which usually last 30 to 60 days: (1) the application is taken, (2) the rate is locked, (3) the loan is funded, and (4) the loan is delivered into the mortgage pool.

In the application stage, the loan is classified as either fixed rate (committed pipeline) or floating rate (uncommitted pipeline). The rate could be locked as early as the application stage but must be done at any point before funding. Once the rate is locked, it is classified as fixed rate and forms part of the committed pipeline.

The period between application and funding is required for the lenders to manage their pipeline. Activities here include underwriting the loan and preparing the related paperwork (e.g., appraisals, credit analysis, title searches, title registration). At this point, the lender is able to sell its loans for settlement in the future (i.e., forward transaction). The lag also allows lenders to control and hedge their production pipeline to meet their profit objectives. Hedging requires lenders to be aware of the rates of new loan applications, as well as those applications that do not finalize (fallout)—steps 1 and 2 complete but not step 3. Generally, fallout will increase significantly if interest rates fall, as the potential borrowers will allow their loan applications to lapse as they apply for other loans at lower rates.

TRADING PASS-THROUGH SECURITIES

AIM 14.4: Discuss MBS market structure and the ways that fixed rate pass-through securities trade.

Trade settlements occur every month on a predetermined basis; delivery dates during a month are specified. In addition, prices are usually quoted for three settlements months, however, trades could be done for a longer period into the future.

Fixed rate pass-through securities trade in one of the following ways:

- Pre-identified pool(s) trade.
- To-be-announced (TBA) trade.
- Stipulated trade.

The pre-identified pool trade identifies the number and balance(s) of the pool(s) prior to trade.

The TBA trade involves identifying the security and establishing the price. However, there is a pool allocation process whereby the actual pools are not revealed to the seller until immediately before settlement. The characteristics of the pools that can be used for TBA trades are regulated to ensure reasonable consistency.

The stipulated trade is similar to the TBA trade, except the pool characteristics are provided in greater detail.
Dollar Roll Transaction

AIM 14.5: Explain a dollar roll transaction, how to value a dollar roll, and what factors can cause a roll to trade “special”.

MBS trading requires the same securities to be priced for different settlement dates. A dollar roll transaction occurs when an MBS market maker buys positions for one settlement month and, at the same time, sells those same positions for another month.

How to Value a Dollar Roll

The process involves assessing the income and the expenses related over the holding period. Income is determined by coupon payments, reinvested interest, and principal payments. Expenses are determined by financing costs [i.e., repurchase (repo) market]. One could purchase the security in the earlier (front) month, hold it, and then dispose of it in the later (back) month at settlement.

The back month price of a dollar roll should take both income and expenses into account so that the net cash flows are equivalent to simply purchasing the security in the back month for settlement at that time. However, empirical evidence suggests that the most likely outcome is that a price drop between the two settlement dates makes purchasing the security in the back month more attractive. In other words, purchasing a position for back month settlement results in financing at an implied repo rate lower than that of the repurchase market.

Factors that impact dollar roll valuations:
* The security’s coupon, age, and WAC.
* Holding period (period between the two settlement dates).
* Assumed prepayment speed.
* Funding cost in the repo market.

Factors Causing a Dollar Roll to Trade Special

When the price difference/drop is large enough to result in financing at less than the implied cost of funds, then the dollar roll is trading special. It could be caused by:
* Decrease in the back month price (due to an increased number of sale/settlement transactions on the back month date by originators).
* Increase in the front month price (due to an increased demand in the front month for deal collateral).
* Shortages of certain securities in the market that require the dealer to suddenly purchase the security for delivery in the front month, thereby increasing the front month price.
Pricing Mortgage Products

AIM 14.6: Relate the pricing of mortgage products to developments in MBS markets.

There has been an increase in consistency of mortgage loan pricing to the consumer as a result of the growth of the MBS sector. The following examples illustrate this point.

In setting the optimal coupon rate for a fixed-rate MBS, the originator wishes to maximize proceeds. The optimal coupon depends on the pass-through prices of different coupons, base servicing fees, excess servicing, guarantee fees, and origination fees.

Mortgage loans also involve points, which are up-front loan fees paid by the borrower. Similar to setting the coupon rate, points are based on the same factors.

Calculating the optimal coupon for jumbo loans (private label) requires the valuation of the cost of credit enhancement (usually subordination). Therefore, the weighted average price of the subordinate classes is considered.

Calculating the optimal coupon for jumbo securitizations (private label) requires the price of the senior securities, the weighted average price of the subordinate bonds, the WAC IO (for premium loans), and the WAC PO (for discount loans).

With the securitization of Alternative-A (Alt-A) loans, there is a choice between GSE and private label. One must consider the price of the senior pass-through and the GSE guarantee fee and compare it to the cost of credit enhancement through private label.

MBS Cash Flow Structuring for Investors

AIM 14.7: Explain the purpose of cash flow structuring of mortgage-backed securities.

There are many different MBS investors (e.g., banks, life insurance companies, pension funds, portfolio managers, hedge funds) who invest in MBSs for different reasons and may have particular cash flow needs. For example, banks may require shorter-term securities, and life insurance companies may require longer-term securities.

The flexible nature of MBSs allows them to be used by portfolio managers in designing specialized investment structures to generate very specific cash flows. For example, it may be necessary to derive a consistent cash flow stream that could be replicated simply by using an IO strip.

Consider a situation where a mortgage bond makes only the scheduled principal payments to its investors (i.e., no prepayments to those investors). Although there may be certainty and stability in those cash flows, it will result in some uncertainty being transferred to the other investors (i.e., may receive prepayments) and more cash flow variability.

As a result, structuring MBSs needs to carefully examine the tradeoffs so as to meet the needs of many investors.
Key Concepts

1. Prior to the existence of the MBS market, financial institutions accepted deposits and then loaned them out. A shortage of local deposit funds could effectively eliminate the opportunity to issue mortgage loans. With the emergence of the MBS market, many mortgage loans with similar characteristics are pooled together to create securities that can be traded. The process essentially transformed what was once a small and relatively illiquid asset into a large and liquid one.

2. There are many types of MBSs to consider, including:
   - Agency fixed-rate pass-throughs.
   - Agency ARM pass-throughs.
   - Private label pass-throughs.
   - Agency mortgage strips.
   - Private label mortgage strips.
   - Agency CMOs.
   - Private label CMOs.

3. There are four steps involved in MBS creation, which usually lasts 30 to 60 days: (1) application taken, (2) rate locked, (3) loan funded, and (4) loan delivered into the pool.

4. Fixed-rate pass-through securities trade in one of the following ways:
   - Pre-identified pool(s) trade.
   - To-be-announced (TBA) trade.
   - Stipulated trade.

5. A dollar roll transaction occurs when an MBS market maker is buying positions for one settlement month and, at the same time, selling those same positions for another month.

6. There has been an increase in consistency of mortgage loan pricing to the consumer as a result of the growth of the MBS sector.

7. There are many different MBS investors (e.g., banks, life insurance companies, pension funds, portfolio managers, hedge funds) who invest in MBSs for different reasons and have particular cash flow needs.
1. In analyzing an agency fixed-rate MBS, the base servicing fees are 25 bp, the guarantee fees are 15 bp, and loan rate is 4.5%, which is 0.7% higher than the pass-through pool coupon rate. Which of the following amounts represents the amount of excess servicing?
   A. 0.1%.
   B. 0.3%.
   C. 0.4%.
   D. 0.7%.

2. Which of the following statements regarding the MBS market is correct?
   A. The MBS market began its development in the early 1990s.
   B. Proceeds from the sales of MBSs are used for additional mortgage lending.
   C. The MBS market has grown significantly over the years and is now almost as large as the Treasury market.
   D. The creation of MBSs is flexible in nature, thereby allowing financial institutions to pool together mortgage loans with a wide range of characteristics.

3. Which of the following statements regarding private label pools and pass-throughs is correct?
   A. Overcollateralization is used especially for prime loans.
   B. The senior tranches will trade at higher yields than the subordinate tranches.
   C. The MBS issuer has discretion in determining the relative sizes of the subordinated tranches required for the deal.
   D. Under a shifting interest structure, it is possible for subordinate tranches to receive cash flows from mortgage prepayments.

4. Which of the following items is not a type of fixed-rate pass-through security trade?
   A. TBA trade.
   B. Stipulated trade.
   C. Mandated trade.
   D. Pre-identified pool trade.

5. Which of the following factors would not cause a dollar roll to trade special?
   A. Decrease in the back month price.
   B. Increase in the front month price.
   C. Surplus of securities in the market used for settlement.
   D. Shortage of securities in the market used for settlement.
1. B Excess servicing = loan rate – base servicing fees – guarantee fees – pass-through pool coupon rate = 4.5% – 0.25% – 0.15% – 3.8% = 0.3%.

2. B The MBS market began its development in the early 1980s. The MBS market has now exceeded the size of the Treasury market. The creation of MBSs requires many mortgage loans with similar characteristics (e.g., interest rate, term, credit quality, loan balance) to be pooled together. The proceeds from the sales of MBSs are used for additional mortgage lending.

3. D Overcollateralization is used when default risk is higher, so it would be more applicable for subprime loans. Given that the senior tranches are less risky, they will trade at lower yields than the subordinate tranches. The rating agencies determine the relative sizes of the subordinated tranches required for the deal. Eventually, the subordinate tranches will receive pro rata shares of prepayments under a shifting interest structure.

4. C There is no such thing as a mandated trade per se. All of the other answer choices are valid types of trades.

5. C When the drop is large enough to result in financing at less than the implied cost of funds, then the dollar roll is trading special. It could be caused by:
   - Decrease in the back month price (due to an increased number of sale/settlement transactions on the back month date by originators).
   - Increase in the front month price (due to an increased demand in the front month for deal collateral).
   - Shortages of certain securities in the market require the dealer to suddenly purchase the security for delivery in the front month, which would increase the front month price.
The following is a review of the Market Risk Measurement and Management principles designed to address the AIM statements set forth by GARP. This topic is also covered in:

**TECHNIQUES FOR VALUING MBSs**

**Exam Focus**

Monte Carlo simulation is the most common methodology used for valuing mortgage-backed securities (MBSs). This technique is able to consider the prepayment risk that is associated with MBSs. Alternate interest rate paths are assumed in the model to generate an option-adjusted spread (OAS). The OAS is superior to static cash flow yield measures such as the nominal spread and the zero-volatility spread or Z-spread. There are, however, limitations associated with the OAS since it is subject to all the modeling risks of Monte Carlo simulations and its underlying assumptions regarding interest rates and prepayments. For the exam, you should understand the calculations, limitations, and advantages of these spreads and be prepared to discuss the steps involved in valuing an MBS using the Monte Carlo methodology.

**Static Valuation**

AIM 15.1: Calculate the static cash flow yield of a MBS using bond equivalent yield (BEY) and determine the associated nominal spread.

AIM 15.2: Define reinvestment risk.

The conventional practice for estimating the yield of an MBS is to calculate a static cash flow yield and compare it to the yield of a Treasury coupon security. The nominal spread is the difference between an MBS static cash flow yield and a Treasury security with the same maturity as the average life of the MBS.

MBSs have monthly cash flow payments, and Treasury securities have semiannual cash flow payments. The yield of Treasury securities is computed by doubling the semiannual yield. However, investors of MBSs must calculate yields on a bond equivalent yield (BEY) basis because compounding occurs more frequently with monthly payments. The BEY for an MBS is calculated as follows:

\[ BEY = 2[(1 + i_M)^6 - 1] \]

where:

\[ i_M = \text{monthly mortgage yield} \]

The monthly mortgage yield is the monthly interest rate that will equate the present value of future monthly cash flows with the market price of the MBS.

Suppose an investor has a mortgage-backed security (MBS) with an average life of ten years. The MBS has a monthly mortgage yield of 0.5%. If a 10-year Treasury bond has a yield of
4.5%, the bond-equivalent yield (BEY) and nominal spread for this MBS are calculated as follows:

\[
BEY = 2[(1 + i_M)^6 - 1] = 2[(1 + 0.005)^6 - 1] = 6.08%
\]

nominal spread = 6.08% - 4.50% = 1.58%

In order to calculate a yield on a security, all future cash flows must be known. This creates a problem for calculating yields of mortgage-backed securities (MBSs) because the rate of prepayment is unknown. Another problem associated with calculating yields for MBSs is the uncertainty regarding future market interest rates. Furthermore, in order to actually realize a yield to maturity (YTM), an investor must reinvest cash flow payments at a rate equal to YTM and hold the security to maturity. For example, the reinvestment of interest or coupon payments has a dramatic impact on the realized return for long-term bonds.

Reinvestment risk is the risk of having to reinvest coupon payments at a lower rate due to decreases in market interest rates. Reinvestment risk is more of a concern with MBSs than bonds. This is because cash flows for MBSs occur monthly, consist of both principal and interest, and are subject to higher prepayment rates. Bonds typically have less frequent semiannual payments that consist of interest only.

**Dynamic Valuation**

AIM 15.3: Describe how the binomial and Monte Carlo valuation methodologies are used for MBS.

Mortgage borrowers have an option to prepay the underlying securities. The value of mortgage-backed securities (MBSs) with embedded options to prepay cannot be determined using traditional option valuation techniques. Therefore, the Monte Carlo valuation methodology is used to value MBSs and other fixed-income securities with embedded options.

The binomial model is only applicable for securities where the decision to exercise a call option is not dependent on how interest rates evolve over time. While the binomial model is useful for callable agency debentures and corporate bonds, it is not applicable to valuing an MBS. The historical evolution of interest rates over time impacts prepayments and makes the binomial model inappropriate for MBSs.

Prepayments on mortgage pass-through securities are interest rate path-dependent. This means that a given month's prepayment rate depends on whether there were prior opportunities to refinance since the origination of the underlying mortgages. For example, if mortgage rates trend downward over a period of time, prepayment rates will increase at the beginning of the trend as homeowners refinance their mortgages, but prepayments will slow as the trend continues because many of the homeowners that can refinance will have already done so. This prepayment pattern is called refinancing burnout. Another problem of the path-dependency of MBSs is related to the nature of structured securities such as collateralized mortgage obligations (CMOs). The amount a CMO tranche receives in
the form of cash flows for a specific month depends on the outstanding balances of other tranches in the deal. These outstanding balances are impacted by earlier principal and interest prepayments.

The **Monte Carlo methodology** is a simulation approach for valuing MBSs. Monte Carlo is actually a process of steps rather than a specific model. It is extremely useful when there are numerous variables with multiple outcomes. Monte Carlo is used to provide a probability distribution of the value of an MBS. The valuation of an MBS is influenced by future interest rates, the shape of the yield curve, future interest rate volatility, prepayment rates, default rates, and recovery rates.

Each of these variables or parameters of the Monte Carlo model could have multiple outcomes with different probabilities associated for each outcome. One valuation approach in these circumstances is the **best guess approach** where the expected value of each variable is used to estimate the value of the MBS. Unfortunately, this method is highly inaccurate. For example, suppose the probability of the best guess occurring for each variable is 70%. Then with six different variables, the probability that the best guess MBS value will occur is only 11.8% (= 0.70^6).

The Monte Carlo approach provides a range of possible outcomes with a probability distribution for the value of a mortgage security. The mean or average value of this range of outcomes is then taken as the estimated value of the MBS. The other information, such as the range of possible outcomes and percentile information, is useful in gauging the value of the security.

---

**AIM 15.4**: Discuss the steps for valuing a mortgage security using Monte Carlo methodology.

The following steps are required to value a mortgage security using the Monte Carlo methodology:

- **Step 1**: Simulate the interest rate path and refinancing path.
- **Step 2**: Project cash flows for each interest rate path.
- **Step 3**: Calculate the present value of cash flows for each interest rate path.
- **Step 4**: Calculate the theoretical value of the mortgage security.

**Step 1**: Simulate the interest rate path and refinancing path.

The first step in applying the Monte Carlo approach is to estimate monthly interest rates for the entire life of the mortgage security. For example, a 30-year mortgage security would require 360 monthly interest rates. In equations to follow, the total number of months on an interest rate path will be denoted by $T$. Also, the total number of interest rate paths or *trials* that are simulated will be denoted by $N$. Random interest rate paths are generated using the term structure of interest rates and a volatility assumption. The term structure of interest rates is created using the theoretical spot rate (zero-coupon) curve for the market on the pricing date. The simulations are adjusted to ensure the average simulated price of a zero-coupon Treasury bond is equal to the actual price corresponding to the pricing date. Some models use LIBOR or swap rates instead of Treasury rates.
The dispersion of future interest rates in the simulation is determined by the volatility assumption. It is common practice to use more than one level of volatility. For example, with a short/long yield volatility approach, the volatility is specified based on maturities. One volatility number is used for shorter maturities (short yield volatility), and a second yield volatility is specified for longer maturities (long yield volatility). Short yield volatility is typically assumed to be greater than long yield volatility. When yield volatility is assumed for each maturity, it is referred to as term structure yield volatility.

The derivatives market is used to construct an arbitrage-free term structure of future interest rates. Short-term interest rate paths are used to discount the cash flows in Step 3 of the Monte Carlo process. These interest rate paths are also used to create the prepayment paths or vectors, which are cash flows for each interest rate path. The prepayment vector is computed based on refinancing rates that are available each month. The mortgagor has an incentive to refinance if the refinancing rate is low relative to the mortgagor’s original coupon rate. The relationship between refinancing rates and short-term interest rates is an important assumption of the model.

Step 2: Project cash flows for each interest rate path.

Cash flows for each month on each interest rate path are equal to the scheduled principal for the mortgage pool, the net interest, and prepayments. Scheduled principal payments are simply calculated based on the projected mortgage balance from the prior month. A prepayment model is used rather than a simple prepayment rate. A prepayment rate is specified for each month on a given interest rate path, and rates for a given month across all interest rate paths are not the same. In fact, there could actually be $T \times N$ different prepayment rates.

CMO deal structures dictate how principal and interest is to be paid. Therefore, it is necessary to reverse engineer the deal to determine the cash flows for a senior CMO. The cash flows for each month on an interest rate path are calculated using the scheduled principal, net interest, and prepayments for the collateral (i.e., the pool of agency pass-throughs). The tranche’s cash flows for each path are determined by the total principal and interest paid to the tranche, the interaction of the cash flow rules, and the prepayment model.

Step 3: Calculate the present value of cash flows for each interest rate path.

The present values of cash flows for each interest rate path are calculated by discounting the cash flows for each path by a discount rate. The discount rate is estimated using the simulated spot rates for each month on the interest rate path plus an appropriate spread. The simulated spot rates are determined from the simulated future monthly rates. The following equation quantifies the relationship that holds between the simulated spot rate, $z_T(n)$, for month $T$ on path $n$, and the simulated future monthly rates, $f_j(n)$:

$$z_T(n) = [(1 + f_1(n))(1 + f_2(n)) \ldots (1 + f_T(n))]^{1/T} - 1$$

where:

$z_T(n)$ = simulated spot rate for month $T$ on path $n$

$f_j(n)$ = simulated future 1-month rate for month $j$ on path $n$

The interest rate paths for the simulated future 1-month rates are converted to the interest rate paths for the simulated monthly spot rates. The present value of the cash flows for
month $T$ on interest rate path $n$ discounted at the simulated spot rate for month $T$, $z_T(n)$, plus a spread, $K$, is:

$$PV[C_T(n)] = \frac{C_T(n)}{[1 + z_T(n) + K]^T}$$

where:
- $PV[C_T(n)]$ = present value of cash flows for month $T$ on path $n$
- $C_T(n)$ = cash flow for month $T$ on path $n$
- $z_T(n)$ = spot rate for month $T$ on path $n$
- $K$ = spread

The present value for path $n$ is determined as the sum of the present values of the cash flows for each month on path $n$ as follows:

$$PV[path(n)] = PV[C_1(n)] + PV[C_2(n)] + ... + PV[C_T(n)]$$

where:
- $PV[path(n)]$ = present value of interest rate path $n$

**Step 4**: Calculate the theoretical value of the mortgage security.

The theoretical value for a specific interest rate path is thought of as the present value of all cash flows in that path, assuming that path was actually realized. The theoretical value of the mortgage security is calculated as the average present value of all theoretical values for each interest rate path as follows:

$$\text{theoretical value} = \frac{PV[path(1)] + PV[path(2)] + ... + PV[path(N)]}{N}$$

where:
- $N$ = number of interest rate paths

This average theoretical value is typically the only measurement that is evaluated when Monte Carlo simulations are used to value MBSs. It is unfortunate that other potentially valuable information, such as the distribution of the path present values, is usually ignored.

**Option-Adjusted Spread**

AIM 15.5: Define and interpret option-adjusted spread (OAS), zero-volatility, and option cost.

The option-adjusted spread (OAS) is defined as the spread, $K$, that, when added to all the spot rates of all the interest rate paths, will make the average present value of the paths
equal to the actual observed market price plus accrued interest. The OAS is mathematically determined by the following relationship:

\[
\text{market price} = \frac{\text{PV}[\text{path}(1)] + \text{PV}[\text{path}(2)] + \ldots + \text{PV}[\text{path}(N)]}{N}
\]

where:

\(N = \text{number of interest rate paths}\)

The left-hand side of the equation is the current market price of the MBS. The right-hand side of the equation is the Monte Carlo model's output of the average theoretical value of the MBS. The OAS is determined with an iterative process. If the average theoretical value determined by the model is higher (lower) than the MBS market value, the spread is increased (decreased).

The OAS can be interpreted as a measure of MBS returns that indicates the potential compensation after adjusting for prepayment risk. In other words, the OAS is option adjusted because the cash flows on the interest rate paths take into account the borrowers' option to prepay. An investor could estimate the value of a security using the OAS for comparable bonds to determine whether or not to invest in the security. A second approach is to compare the OAS generated at the market price to those available for comparable securities or an investment benchmark (such as a cost of funds).

The OAS is superior to the nominal spread. The nominal spread is calculated as the difference of a mortgage-backed security (MBS) bond-equivalent yield (BEY) and a comparable Treasury security. The nominal spread is a static cash flow measure that is the most commonly quoted measure of MBS incremental returns. However, the nominal spread does not consider prepayment risk that is associated with support tranches. Therefore, the nominal spread does not indicate if the investor is appropriately compensated for prepayment risk. Another problem with the nominal spread is the difference in timing of cash flows between MBSs and Treasury securities.

Cash flows for MBSs are monthly annuity payments, while Treasury securities pay semiannual interest-only payments and a large bullet payment. The zero-volatility spread (sometimes referred to as the Z-spread) is a spread measure that an investor realizes over the entire Treasury spot rate curve, assuming the mortgage security is held to maturity. It is a more accurate measure because it compares an MBS to a portfolio of Treasury securities. The zero-volatility spread is the yield that equates the present value of the cash flows from the MBS to the price of the MBS discounted at the Treasury spot rate plus the spread. Thus, an iterative process is required to determine the zero-volatility spread.

The zero-volatility spread accounts for variations in MBS principal payments at a given prepayment rate or speed. However, it does not consider the impact that prepayment risk or changing prepayment rates have on the value of the MBS.

The option cost measures the prepayment (or option) risk. It is the implied cost of the option embedded in the MBS. The option cost is calculated as the difference between the OAS at the assumed volatility of interest rates and the zero-volatility spread as follows:

\[
\text{option cost} = \text{zero-volatility spread} - \text{OAS}
\]
Therefore, the option cost is a by-product of the Monte Carlo analysis and is not
determined using traditional option value approaches. As volatility declines, the option cost
decreases, and the previously described relationship suggests that OAS increases as volatility
declines, all other things equal.

**Increasing the Number of Rate Paths**

AIM 15.6: Explain how to select the number of interest rate paths in Monte Carlo analysis.

A better estimate of the theoretical value of an MBS is obtained by increasing the number
of simulated interest rate paths. The most common measurement for determining how good
an estimate is from a Monte Carlo simulation model is the **mean standard error** (MSE),
which is defined as:

\[
MSE = \sqrt{\frac{\text{variance of the trial values}}{\text{number of trials}}}
\]

A smaller MSE indicates a better estimate of the output value from the Monte Carlo
simulation. Increasing the number of trials will reduce MSE. However, there are costs
associated with increasing the number of trials.

Alternatively, some literature suggests a variance reduction approach that involves reducing
the variance of the trial values. A **variance reduction technique** is used by most Monte
Carlo models to reduce the number of simulated paths required to get a good statistical
sample. Variance reduction techniques result in price estimates within 1/32nd of a point
(i.e., a tick). A typical analysis would be between 256 and 1,024 interest rate paths. If 1,024
paths are used to obtain the estimated price for a tranche, there is little benefit gained by
increasing the number of trials.

Some vendors have also reduced the number of paths required using a statistical technique
that creates representative paths. This technique combines similar sets of paths into
representative paths. Principal component analysis is a statistical technique that may reduce
2,000 sample paths down to 16 representative paths. Each representative path is then
assigned a weight relative to the total sample of paths. This technique is mathematically
expressed as:

\[
\text{theoretical value} = W_1PV[Rpath(1)] + W_2PV[Rpath(2)] + \ldots + W_JPV[Rpath(J)]
\]

where:

- \( J \) = number of representative paths
- \( W_j \) = number of sample interest rate paths represented by representative path \( j \)
  divided by the total number of sample interest rate paths
- \( Rpath(j) \) = representative path \( j \)

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**Total Return Analysis**

AIM 15.7: Describe total return analysis, calculate total return, and understand factors present in more sophisticated models.

Total return analysis is a technique used by investors to evaluate returns for different horizons and interest rate scenarios. One advantage of using total return analysis is it allows investors to specify reinvestment returns. The following parameters are required to calculate the total return for a mortgage-backed security (MBS):

- The initial security cost at the time of purchase.
- Projected cash flows for the security, including scheduled and unscheduled principal payments, interest, and reinvestment income.
- The security's projected horizon value at the horizon date.

The periodic total percentage return over the time horizon is calculated as:

\[
\text{periodic total return} = \frac{\text{total horizon proceeds}}{\text{total cost}} - 1
\]

The total percentage return can then be annualized for time horizons less than 12 months using the following formula:

\[
\text{annualized total return} = \left( \frac{\text{periodic total return}}{12} \right)^{\frac{12}{\text{number of months in period}}}
\]

**Example: Calculating total return for an MBS**

Calculate the total periodic return and the annualized total return for a Fannie Mae 6% pass-through over a 6-month horizon with the information provided in Figure 1.
Figure 1: Pass-Through Fannie Mae Security Information

<table>
<thead>
<tr>
<th>Current settlement investment balance</th>
<th>10,000,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial price</td>
<td>100 5/16</td>
</tr>
<tr>
<td>Horizon price</td>
<td>100 5/8</td>
</tr>
<tr>
<td>Purchase settlement date</td>
<td>10/15/2010</td>
</tr>
<tr>
<td>Horizon settlement date</td>
<td>4/15/2011</td>
</tr>
<tr>
<td>Face value × price</td>
<td>10,031,250</td>
</tr>
<tr>
<td>Accrued interest</td>
<td>18,333</td>
</tr>
<tr>
<td>Total cost</td>
<td>10,049,583</td>
</tr>
<tr>
<td>Principal received (par value)</td>
<td>1,104,904</td>
</tr>
<tr>
<td>Interest received</td>
<td>285,879</td>
</tr>
<tr>
<td>Reinvestment income</td>
<td>12,239</td>
</tr>
<tr>
<td>Total interim cash flows</td>
<td>1,403,022</td>
</tr>
<tr>
<td>Dollar value of remaining principal</td>
<td>8,950,690</td>
</tr>
<tr>
<td>Accrued interest</td>
<td>16,308</td>
</tr>
<tr>
<td>Total horizon value</td>
<td>8,966,998</td>
</tr>
</tbody>
</table>

**Answer:**

The numerator for calculating the periodic total return is the total horizon proceeds (dollar value of remaining principal plus accrued interest) and interim cash flows, which include interest paid to the investor on the declining monthly balance, the principal paid back to the investor (valued at par), and reinvestment income at the specified reinvestment rate. The denominator is simply the total cost of the security, which consists of the original dollar price and accrued interest. The periodic total percentage return over the time horizon is calculated as:

\[
\text{periodic total return} = \frac{1,403,022 + 8,966,998}{10,049,583} - 1 = 3.189\%
\]

The total percentage return is annualized by multiplying the periodic total return by 12 divided by 6 (the monthly time horizon for the MBS):

\[
\text{annualized total return} = (0.03189)^\frac{12}{6} = 6.377\%
\]
Total return models are usually more sophisticated than the simple calculations shown in the preceding example. In more sophisticated models, returns are generated in multiple interest rate scenarios that assume both parallel and nonparallel interest rate shifts. Models also generate additional scenarios that examine changes in implied volatilities.

Additional scenarios are generated by total return models that use valuation and prepayment models. Base case and multiple scenarios adjusting prepayments are used to generate horizon prices under varying assumptions. For example, option-adjusted spreads can be simulated from multiple interest rate paths. Alternatively, historical data could be used to estimate spreads for an MBS using different interest rate levels. These spreads are then used to generate horizon prices for various interest rate scenarios.

Numerous factors are important to consider for return calculations of MBSs. The reinvestment and prepayment assumptions are extremely important for certain types of MBSs. The cash flows of MBSs are path dependent, as changes in interest rates will impact prepayments and reinvestment. Most models adjust shifts in interest rates either immediately, gradually over time, or at the horizon. In addition to calculating total return, total return models also consider other calculations (e.g., cash profit and loss, return on equity, and financing-adjusted returns).

**LIMITATIONS OF MBS VALUATION MEASURES**

AIM 15.8: Discuss limitations of the nominal spread, Z-spread, OAS, and total return measures.

The nominal spread does not consider prepayment risk that is associated with support tranches and, therefore, does not indicate if the investor is appropriately compensated for prepayment risk. Another problem with the nominal spread is the difference in timing of cash flows for the MBS and Treasury securities. Cash flows for MBSs are monthly annuity payments, while Treasury securities pay semiannual interest-only payments and a large bullet payment.

The Z-spread is a measure that an investor realizes over the entire Treasury spot rate curve, assuming the mortgage security is held to maturity. It is a more accurate measure than the nominal spread because it compares an MBS to a portfolio of Treasury securities. The Z-spread accounts for variations in MBS principal payments at a given prepayment rate or speed. However, it does not consider the impact that prepayment risk or changing prepayment rates have on the value of the MBS.

Option-adjusted spreads (OASs) are a superior relative measure of value compared to static cash flow measures. However, there are four important limitations to consider when using this measure. The four major limitations of OASs are related to:

- Modeling risk associated with Monte Carlo simulations.
- Required adjustments to interest rate paths.
- An underlying assumption of a constant OAS over time in the model.
- The dependency of the underlying prepayment model.

The OAS is generated through Monte Carlo simulations. Therefore, the OAS is subject to all modeling risks associated with the simulation. Interest rate paths must be adjusted to
ensure securities or rates making up the benchmark curve are properly valued when using Monte Carlo methods. This process of adjusting interest rate paths is subject to modeling error. If there is a term structure to the OAS, then this is not reflected in the Monte Carlo process because the OAS methodology assumes a constant OAS.

The prepayment model is very complex, given the amount of uncertainty regarding important variables. The behavior of both borrowers and lenders changes over time. Thus, the greatest weakness of using OAS valuation estimates generated from the Monte Carlo simulation is the dependence on the prepayment model.

Nominal spreads, Z-spreads, and OASs all have one major limitation in common. All of these spread measures assume the securities are held to maturity. Some investors may hold a security to maturity, but many investors will only hold a security over a finite horizon. Thus, the investor should analyze the securities in a manner that is consistent with the investor’s asset management horizon.

While the total return model allows the investor to analyze securities for a finite horizon, it is still subject to other limitations. The total return model’s output is very dependent on assumptions regarding the horizon price and prepayment rates. Therefore, numerous scenarios and greater flexibility of more complex return models can be a disadvantage because there is a great deal of uncertainty regarding horizon values and prepayment rates. Another limitation is that dynamic rate scenarios are usually not included in the total return models.
KEY CONCEPTS

1. The nominal spread is the difference between an MBS static cash flow yield (calculated as a BEY) and a Treasury security with the same maturity as the average life of the MBS. The BEY for an MBS is calculated as follows: \( \text{BEY} = 2 \left( 1 + i_M \right)^{12} - 1 \), where \( i_M \) is the monthly mortgage yield. Reinvestment risk is the risk of having to reinvest coupon payments at a lower rate due to decreases in market interest rates.

2. The Monte Carlo methodology is a simulation approach for valuing MBSs. The binomial model is not appropriate for valuing MBSs because MBSs have embedded prepayment options, and the historical evolution of interest rates over time impacts prepayments.

3. A mortgage security is valued using the Monte Carlo methodology by simulating the interest rate path and refinancing path, projecting cash flows for each interest rate path, calculating the present value of cash flows for each interest rate path, and calculating the theoretical value of the mortgage security.

4. The option-adjusted spread (OAS) is the spread that, when added to all the spot rates of all the interest rate paths, will make the average present value of the paths equal to the actual observed market price plus accrued interest. The zero-volatility spread (Z-spread) is the spread that an investor realizes over the entire Treasury spot rate curve, assuming the mortgage security is held to maturity. The option cost is the implied cost of the embedded prepayment option and is calculated as the difference between the Z-spread and OAS.

5. The most common approach to determine the number of simulated interest rate paths used in Monte Carlo is based on minimizing the mean standard error:

\[
\text{MSE} = \sqrt{\frac{\text{variance of the trial values}}{\text{number of trials}}}
\]

6. Total return analysis is a useful technique when evaluating returns for different horizon and interest rate scenarios. The total return for an MBS is calculated using the initial security cost at the time of purchase, projected interim cash flows for the security, and the security's projected horizon value.

7. The nominal spread and Z-spread do not consider prepayment risks. Thus, option-adjusted spreads (OASs) are a superior relative measure of value because they account for prepayment risks. However, four major limitations of OASs are related to modeling risk associated with Monte Carlo simulations; required adjustments to interest rate paths; model assumption of a constant OAS over time; and dependency on the underlying prepayment model.
1. Suppose an investor has a mortgage-backed security (MBS) with a monthly mortgage yield of 0.3%. What is the bond-equivalent yield (BEY) for this MBS?
   A. 1.02%.
   B. 1.80%.
   C. 3.63%.
   D. 6.07%.

2. The binomial approach is not appropriate for valuing mortgage-backed securities (MBSs) because:
   A. interest rates fluctuate over time.
   B. interest rates cannot be accurately forecasted.
   C. this approach is not appropriate for valuing securities with embedded options.
   D. this approach requires the Z-spread, which is a static valuation.

3. When using the Monte Carlo approach to estimate the value of mortgage-backed securities (MBSs), the model should:
   A. use one consistent volatility measure for all interest rate paths.
   B. use a short/long yield volatility approach.
   C. use annual interest rates over the entire life of the mortgage security.
   D. ignore the distribution of the interest rate paths used to determine the theoretical value.

4. What is the total annualized return for a 6-month mortgage pass-through security that has an original cost plus accrued interest of $10,065,630, with proceeds that consist of payments of $1,364,798, reinvestment income of $11,054, and a horizon value of $8,970,542?
   A. 2.79%.
   B. 4.51%.
   C. 5.36%.
   D. 5.58%.

5. All of the following describe limitations of using option-adjusted spreads (OASs) for valuing mortgage-backed securities (MBSs) except:
   A. modeling risk is associated with Monte Carlo simulations.
   B. model requires making adjustments to interest rate paths.
   C. model assumes a dynamic OAS over time.
   D. prepayment model influences the model valuation.
**Concept Checker Answers**

1. C The bond-equivalent yield is calculated using the following formula:
   \[ \text{BEY} = 2[(1 + i_J)^6 - 1] = 2[(1 + 0.003)^6 - 1] = 3.63\% \]

2. A The binomial approach is not appropriate for valuing mortgage-backed securities (MBSs) because interest rates fluctuate over time. The value of an MBS is dependent on the historical path of interest rates and the impact of prepayments. The binomial model is only appropriate in valuing securities where the embedded option is not dependent on how interest rates evolve over time.

3. B When using the Monte Carlo approach to estimate the value of mortgage-backed securities (MBSs), the model should use more than one volatility measure for all interest rate paths. It is very common to use a short/long yield volatility approach to estimate monthly rates. Although, the information regarding the distributions of interest rate paths is oftentimes ignored, it contains valuable information and should be considered.

4. D Start by calculating the periodic total percentage return over the 6-month horizon as follows:
   \[ \text{periodic total return} = \frac{1,364,798 + 11,054 + 8,970,542}{10,065,630} - 1 = 2.789\% \]
   
The total percentage return can then be annualized using the following formula:
   \[ \text{annualized total return} = (0.02789)^\frac{12}{6} = 5.579\% \]

5. C When using OAS to value MBS, the model assumes a constant OAS over time. This is problematic if there is a term structure to the OAS because this is not reflected in the Monte Carlo process.
CHALLENGE PROBLEMS

1. You are a quantitative analyst at an insurance company. Given some large losses incurred by the company recently, your boss is interested in determining the expected number of extreme losses per year. As well, your boss is quite certain that the company is now more likely to experience an extreme event than before. Based on the information provided by your boss, to model the frequency and severity of extreme events, which of the following distributions would be most appropriate to use?

<table>
<thead>
<tr>
<th>Frequency</th>
<th>Severity</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Poisson distribution</td>
<td>Frechet distribution</td>
</tr>
<tr>
<td>B. Poisson distribution</td>
<td>Gumbel distribution</td>
</tr>
<tr>
<td>C. Weibull distribution</td>
<td>Frechet distribution</td>
</tr>
<tr>
<td>D. Weibull distribution</td>
<td>Gumbel distribution</td>
</tr>
</tbody>
</table>

2. The generalized Pareto distribution is used for modeling extreme losses. The model requires the choice of a threshold. Which of the following best describes the tradeoffs in setting the threshold level?

A. The threshold must be high enough so that the tail index indicates a heavy tail.
B. The threshold must be high enough so that the tail index indicates a light tail.
C. The threshold must be high enough so that convergence to the generalized Pareto distribution occurs.
D. The threshold must be high enough so that there are enough observations to estimate the parameters.

3. You are a bond portfolio manager and are analyzing the price sensitivity of various potential bond portfolios. Which of the following bond portfolios will have the greatest price sensitivity?

A. Portfolio A: 100% weight in a six year semiannual pay coupon bond.
B. Portfolio B: 100% weight in a six year annual pay coupon bond.
C. Portfolio C: 50% weight in a two year zero coupon bond and 50% weight in a ten year zero coupon bond.
D. Portfolio D: 50% weight in a four year zero coupon bond and 50% weight in an eight year zero coupon bond.

4. A constant maturity Treasury (CMT) swap pays \( (1,000,000 / 2) \times (y_{CMT} - 9\%) \) every six months. There is a 70% probability of an increase in the 6-month spot rate and a 60% probability of an increase in the 1-year spot rate. The rate change in all cases is 0.50% per period, and the initial \( y_{CMT} \) is 9%. What is the value of this CMT swap?

A. $2,325.
B. $2,229.
C. $2,429.
D. $905.
5. Which of the following statements is incorrect regarding volatility smiles?
   A. Currency options exhibit volatility smiles because the at-the-money options have higher implied volatility than away-from-the-money options.
   B. Volatility frowns result when jumps occur in asset prices.
   C. Equity options exhibit a volatility smirk because low strike price options have greater implied volatility.
   D. Relative to currency traders, it appears that equity traders' expectations of extreme price movements are more asymmetric.

6. You are an institutional portfolio manager. One of your clients is very interested in the flexibility of options but expresses great concern about the high cost of some of them. In general, which of the following options would be the least costly to purchase?
   A. Shout options.
   B. American options.
   C. Lookback options.
   D. Bermudan options.

7. You believe that a stock will increase in price and would like to buy a call option. You would like to choose the date during the option's term when the option payoff is determined. However, if the option payoff is greater at the option's maturity, you want to be paid this value. What type of option should you buy?
   A. Chooser option.
   B. Compound option.
   C. Shout option.
   D. Asian option.
CHALLENGE PROBLEM ANSWERS

1. A A Poisson distribution is frequently assumed for the distribution of operational risk event frequency (i.e., number of losses per year). In contrast, a Weibull distribution is used when modeling the severity of operational risk losses. Therefore, the Poisson distribution is the more appropriate one to use to model the distribution of frequency.

A Frechet distribution has “heavy” tails, which suggests that there is a greater likelihood of an extreme event occurring. In contrast, a Gumbel distribution has “light” tails, which suggests that compared to the Frechet distribution, there is a lesser likelihood of an extreme event occurring. Gumbel distributions are similar to normal and lognormal distributions where there is a lesser likelihood of an extreme event occurring. Therefore, the Frechet distribution is the more appropriate one to use to model the distribution of severity.

(See Topic 4)

2. C The threshold must be high enough so that convergence to the generalized Pareto distribution occurs. Choices A and B are incorrect because the tail index is chosen by the researcher. Heavy tails are indicated by a tail index greater than zero. Choice D is incorrect because the threshold must be low enough so that there are enough observations to estimate the parameters.

(See Topic 4)

3. C Price sensitivity increases as duration increases and as convexity increases.

Choice A is incorrect because it will have the shortest duration. Its duration will be less than the six year maturity because it pays a coupon. It pays its coupon semiannually so its duration is shorter than that for Portfolio B.

Choice B is incorrect because it will have the second shortest duration. Its duration will be less than six years because it pays a coupon.

Both Portfolio C and Portfolio D will have durations of six years because portfolio duration is a simple weighted average of component bond durations. The duration for a zero coupon bond is its maturity.

However, Portfolio C will have the higher convexity because Portfolio C contains the longest maturity bond and convexity increases with the square of maturity. So Portfolio C will have the greatest price sensitivity.

(See Topic 7)
4. A The payoff in each period is $(1,000,000 / 2) \times (y_{CMT} - 9\%)$. For example, the 1-year payoff of $5,000 in the figure below is calculated as $(1,000,000 / 2) \times (10\% - 9\%) = 5,000$. The other numbers in the year one cells are calculated similarly.

In six months, the payoff if interest rates increase to 9.50% is $(1,000,000 / 2) \times (9.5\% - 9.0\%) = 2,500$. Note that the price in this cell equals the present value of the probability weighted 1-year values plus the 6-month payoff:

$$V_{6\text{ month,}U} = \frac{(5,000 \times 0.6) + (0 \times 0.4)}{1 + 0.095} + 2,500 = 5,363.96$$

The other cell value in six months is calculated similarly and results in a loss of $4,418.47$.

The value of the CMT swap today is the present value of the probability weighted 6-month values:

$$V_0 = \frac{(5,363.96 \times 0.7) + (-4,418.47 \times 0.3)}{1 + 0.09} = 2,324.62$$

Thus the correct response is A. The other answers are incorrect because they do not correctly discount the future values or omit the 6-month payoff from the 6-month values.

(See Topic 9)

5. A Currency options exhibit volatility smiles because the at-the-money options have lower implied volatility than away-from-the-money options.

Equity traders believe that the probability of large price decreases is greater than the probability of large price increases. Currency traders' beliefs about volatility are more symmetric as there is no large skew in the distribution of expected currency values (i.e., there is a greater chance of large price movements in either direction).

(See Topic 10)
6. D  Bermudan options may be exercised early (like American options) but exercise is restricted to certain dates. Therefore, the restriction suggests that Bermudan options must be cheaper than American options.

(See Topic 11)

7. C  The shout option allows the buyer to choose the date when he “shouts” to the option seller that the intrinsic value should be determined. At expiration, the option buyer receives the maximum of the shout value or the intrinsic value at expiration.

(See Topic 11)
GARP FRM Practice Exam Questions

Market Risk Measurement and Management

Professor’s Note: The following questions are from the 2008–2011 GARP FRM Practice Exams.

1. After estimating the 99%, 1-day VaR of a bank’s portfolio to be USD 1,484 using historical simulation with 1,000 past trading days, you are concerned that the VaR measure is not providing enough information about tail losses. You decide to re-examine the simulation results and sort the simulated daily P&L from worst to best giving the following worst 15 scenarios:

<table>
<thead>
<tr>
<th>Scenario Rank</th>
<th>Daily P/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>USD -2,833</td>
</tr>
<tr>
<td>2</td>
<td>USD -2,333</td>
</tr>
<tr>
<td>3</td>
<td>USD -2,228</td>
</tr>
<tr>
<td>4</td>
<td>USD -2,084</td>
</tr>
<tr>
<td>5</td>
<td>USD -1,960</td>
</tr>
<tr>
<td>6</td>
<td>USD -1,751</td>
</tr>
<tr>
<td>7</td>
<td>USD -1,679</td>
</tr>
<tr>
<td>8</td>
<td>USD -1,558</td>
</tr>
<tr>
<td>9</td>
<td>USD -1,542</td>
</tr>
<tr>
<td>10</td>
<td>USD -1,484</td>
</tr>
<tr>
<td>11</td>
<td>USD -1,450</td>
</tr>
<tr>
<td>12</td>
<td>USD -1,428</td>
</tr>
<tr>
<td>13</td>
<td>USD -1,368</td>
</tr>
<tr>
<td>14</td>
<td>USD -1,347</td>
</tr>
<tr>
<td>15</td>
<td>USD -1,319</td>
</tr>
</tbody>
</table>

What is the 99%, 1-day expected shortfall of the portfolio?
A. USD 433
B. USD 1,285
C. USD 1,945
D. USD 2,833

2. A fixed-income portfolio has a market value of USD 60 million, modified duration of 2.53 years and is yielding 4.796 compounded semiannually. What would be the change in the value of this portfolio after a parallel rate decline of 20 basis points in the yield curve?
A. A loss of USD 607,200
B. A loss of USD 303,600
C. A gain of USD 303,600
D. A gain of USD 607,200
3. John Snow's portfolio has a fixed-income position with market value of USD 70 million, modified duration of 6.44 years and is yielding 6.7% compounded semiannually. If there is a positive parallel shift in the yield curve of 25 basis points, which of the following answers best estimates the resulting change in the value of John's portfolio?
   A. USD −11,725
   B. USD −1,127,000
   C. USD −1,134,692
   D. USD −1,164,755

4. A 1-year forward contract on a stock with a forward price of USD 100 is available for USD 1.50. The table below lists the prices of some barrier options on the same stock with a maturity of 1 year and strike of USD 100. Assuming a continuously compounded risk-free rate of 5% per year, what is the price of a European put option on the stock with a strike of USD 100?

<table>
<thead>
<tr>
<th>Option</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Up-and-in barrier call, barrier USD 95</td>
<td>USD 5.21</td>
</tr>
<tr>
<td>Up-and-out barrier call, barrier USD 95</td>
<td>USD 1.40</td>
</tr>
<tr>
<td>Down-and-in barrier put, barrier USD 80</td>
<td>USD 3.5</td>
</tr>
</tbody>
</table>

   A. USD 2.00
   B. USD 4.90
   C. USD 5.11
   D. USD 6.61

5. Assuming equal strike prices and expiration dates, which of the following options should be the least expensive?
   A. American call option
   B. Shout call option
   C. European call option
   D. Lookback call option

6. George Smith is an analyst in the risk management department and he is reviewing a pool of mortgages. Prepayment risk introduces complexity to the valuation of mortgages. Which of the two factors is generally considered to affect prepayment risk for a mortgage?
   I. Changes to interest rates
   II. Amount of principal outstanding

   A. I only
   B. II only
   C. Both
   D. Neither
7. Which of following statements about mortgage-backed securities (MBS) is correct?
   I. The price of a MBS is more sensitive to yield curve twists than zero-coupon bonds.
   II. When the yield is higher than the coupon rate of a MBS, the MBS behaves similar to corporate bonds as interest rates change.
   A. I only
   B. II only
   C. Both
   D. Neither

8. Which of following statement about mortgage-backed securities (MBS) is correct?
   I. As yield volatility increases, the value of a MBS grows as well.
   II. A rise in interest rates increases the duration of a MBS.
   A. I only
   B. II only
   C. Both
   D. Neither

9. A market risk manager uses historical information on 1,000 days of profit/loss information to calculate a daily VaR at the 99th percentile, of USD 8 million. Loss observations beyond the 99th percentile are then used to estimate the conditional VaR. If the losses beyond the VaR level, in millions, are USD 9, USD 10, USD 11, USD 13, USD 15, USD 18, USD 21, USD 24, and USD 32, then what is the conditional VaR?
   A. USD 9 million
   B. USD 32 million
   C. USD 15 million
   D. USD 17 million

10. Sarah is a risk manager responsible for the fixed income portfolio of a large insurance company. The portfolio contains a 30-year zero coupon bond issued by the US Treasury (STRIPS) with a 5% yield. What is the bond's DV01?
    A. 0.0161
    B. 0.0665
    C. 0.0692
    D. 0.0694

11. You are asked to mark to market a book of plain vanilla stock options. The trader is short deep out-of-money options and long at-the-money options. There is a pronounced smile for these options. The trader’s bonus increases as the value of his book increases. Which approach should you use to mark the book?
    A. Use the implied volatility of at-the-money options because the estimation of the volatility is more reliable.
    B. Use the average of the implied volatilities for the traded options for which you have data because all options should have the same implied volatility with Black-Scholes and you don't know which one is the right one.
    C. For each option, use the implied volatility of the most similar option traded on the market.
    D. Use the historical volatility because doing so corrects for the pricing mistakes in the option market.
12. Looking at a risk report, Mr. Woo finds that the options book of Ms. Yu has only long positions and yet has a negative delta. He asks you to explain how that is possible. What is a possible explanation?
   A. The book has a long position in up-and-in call options.
   B. The book has a long position in binary options.
   C. The book has a long position in up-and-out call options.
   D. The book has a long position in down-and-out call options.

13. You have a long position in a digital call option—an option that is also called cash-or-nothing—on shares in Global Enterprises. The digital call has a strike price of USD 20 with one year remaining to expiration. Assume that the shares currently trade at USD 22 and annual return volatility of Global Enterprises shares is 15%.
   Which of the following sensitivities would be associated with this option?
   
   I. Delta is positive.
   II. Gamma is positive.
   III. Vega is negative.
   IV. Vega is positive.
   
   Which statements are true?
   A. I and III
   B. IV only
   C. I, II, and IV
   D. II and III

14. Which of the following statements regarding Extreme Value Theory (EVT) is incorrect?
   A. Conventional approaches for estimating VaR that assume that the distribution of returns follow a unique distribution for the entire range of values may fail to properly account for the fat tails of the distribution of returns.
   B. In contrast to conventional approaches for estimating VaR, EVT only considers the tail behavior of the distribution.
   C. By smoothing the tail of the distribution, EVT effectively ignores extreme events and losses which can generally be labeled outliers.
   D. EVT attempts to find the optimal point beyond which all values belong to the tail and then models the distribution of the tail separately.

15. Consider a position in a 5-year receive-fixed swap that makes annual payments on a USD 100 million notional. The floating leg has just been reset. The term structure is flat at 5%, the Macaulay duration of a 5-year par bond is 4.5 years, and the annual volatility of yield changes is 100bp. Your best estimate of the swap's VaR with 95% confidence over the next month is:
   A. USD 1.6 million
   B. USD 2.0 million
   C. USD 5.5 million
   D. USD 7.1 million
16. A bond trader has bought a position in Treasury Bonds with a 4% annual coupon rate on February 15, 2015. The DV01 of the position is USD 80,000. The trader decides to hedge his interest rate risk with the 4.5% coupon rate Treasury Bonds maturing on May 15, 2017 which has a DV01 of .076 per USD 100 face value. To implement this hedge, approximately what face amount of the 4.5% Treasury bonds maturing on May 15, 2017 should the trader sell?
A. USD 80,000  
B. USD 10,500,000  
C. USD 80,000,000  
D. USD 105,000,000

17. The following table shows the composition of the GARP Bond Fund. What are the portfolio duration and portfolio yield of the fund?

<table>
<thead>
<tr>
<th>GARP Bond Fund</th>
<th>Amount in USD</th>
<th>Duration in years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rating</td>
<td></td>
<td></td>
</tr>
<tr>
<td>AAA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Company A</td>
<td>600</td>
<td>1.5</td>
</tr>
<tr>
<td>Company B</td>
<td>300</td>
<td>4</td>
</tr>
<tr>
<td>Company C</td>
<td>200</td>
<td>2.5</td>
</tr>
<tr>
<td>AA</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Company D</td>
<td>400</td>
<td>4</td>
</tr>
<tr>
<td>Company E</td>
<td>350</td>
<td>0.5</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Company F</td>
<td>150</td>
<td>1.5</td>
</tr>
<tr>
<td>Total</td>
<td>2,000</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rating Valuation Matrix</th>
<th>0–1</th>
<th>1–2</th>
<th>2–3</th>
<th>3–4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rating</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>AAA</td>
<td>6.25%</td>
<td>6.75%</td>
<td>7.35%</td>
<td>8.00%</td>
</tr>
<tr>
<td>AA</td>
<td>6.75%</td>
<td>7.35%</td>
<td>8.05%</td>
<td>8.80%</td>
</tr>
<tr>
<td>A</td>
<td>7.75%</td>
<td>8.45%</td>
<td>9.15%</td>
<td>9.85%</td>
</tr>
</tbody>
</table>

A. 14 years, 46.1%  
B. 2.3 years, 7.5%  
C. 2.3 years, 7.7%  
D. 4.4 years, 15.4%

18. Considering options generally (i.e., not only plain vanilla calls and puts), which of the following statements about vega is correct?
A. An option holder can never be vega negative.  
B. A deep in the money up and out call option has a negative vega.  
C. A deep out of money up and out call option has a negative vega.  
D. A deep out of money digital option has a negative vega.
19. Assuming other things constant, bonds of equal maturity will still have different
DV01 per USD 100 Face Value. Their DV01 per USD 100 Face Value will be in the
following sequence of Highest Value to Lowest Value:
A. Zero Coupon Bonds, Par Bonds, Premium Bonds
B. Premium Bonds, Par Bonds, Zero Coupon Bonds
C. Premium Bonds, Zero Coupon Bonds, Par Bonds
D. Zero Coupon Bonds, Premium Bonds, Par Bonds

20. A zero-coupon bond with a maturity of 10 years has an annual effective yield of
10%. What is the closest value for its modified duration?
A. 9
B. 10
C. 100
D. Insufficient Information

21. An analyst observes that the market price of a call option is USD 5 higher than the
theoretical price dictated by an option pricing model. Assuming the same strike
price and time to expiration, what should be the relationship between the market
price of a put option and its theoretical price dictated by the option pricing model?
A. The market price will be higher than the theoretical price by USD 5.
B. The market price will be higher than the theoretical price by an amount lower
than USD 5.
C. The market price will be lower than the theoretical price by USD 5.
D. The market price will be lower than the theoretical price by an amount lower
than USD 5.

22. All else being equal, which of the following options would cost more than plain
vanilla options?
   I. Lookback options
   II. Barrier options
   III. Asian options
   IV. Chooser options
A. I only
B. I and IV
C. II and III
D. I, III and IV

23. Which type of option produces discontinuous payoff profiles, meaning that the
payoff does not increase or decrease continuously with the underlying asset value?
A. Chooser options
B. Barrier options
C. Binary options
D. Lookback options
24. Which one of the following statements on hedging exotic options is incorrect?
   A. Asian options are more difficult to hedge because they have more extreme gamma towards expiration.
   B. Barrier options are more difficult to hedge because delta is liable to be discontinuous at the barrier.
   C. The approach of static options replication is to find a portfolio of regular options whose value matches the value of the exotic option on some boundary.
   D. The portfolio constructed using static option replication must be unwound when any part of the boundary is reached.

25. Which of the following mortgage backed securities has a negative duration?
   A. Interest-only strips (IO)
   B. Inverse floater
   C. Mortgage pass-through
   D. Principal-only strips (PO)

26. Two comparable (same credit rating, maturity, liquidity, rate) U.S. callable corporate bonds are being analyzed by you. The following data is available for the nominal spread over the U.S. Treasury yield curve and Z-spread and option adjusted spread relative to the U.S. Treasury spot curve.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nominal spread</td>
<td>145</td>
<td>130</td>
</tr>
<tr>
<td>Z-spread</td>
<td>120</td>
<td>115</td>
</tr>
<tr>
<td>OAS</td>
<td>100</td>
<td>105</td>
</tr>
</tbody>
</table>

The nominal spread on the comparable option free bonds in the market is 100 basis points. Which of the following statements is correct?
   A. X is undervalued
   B. Y is undervalued
   C. X and Y both are undervalued
   D. Neither X nor Y is undervalued
GARP FRM Practice Exam Answers

Market Risk Measurement and Management

Question from the 2011 FRM practice exam.

1. C USD 1,945

Expected shortfall = average of the worst 10 daily P&L = USD 1945.

(See Topic 1)

Question from the 2011 FRM practice exam.

2. C A gain of USD 303,600

By definition, $D_{mod} = (-1/P) \cdot (dP/dy)$. So as a linear approximation,

$\Delta P = -1 \cdot \Delta y \cdot D_{mod} \cdot P = -1 \cdot 0.0020 \cdot 2.53 \cdot 60 = 0.3036$ million

(See Topic 7)

Question from the 2011 FRM practice exam.

3. B USD -1,127,000

By definition, $D_{mod} = (-1/P) \cdot (dP/dy)$. So as a linear approximation,

$\Delta P = -1 \cdot \Delta y \cdot D_{mod} \cdot P = -1 \cdot 0.0025 \cdot 6.44 \cdot 70 = -1.127$ million

(See Topic 7)

Question from the 2011 FRM practice exam.

4. C USD 5.11

The sum of the price of up-and-in barrier call and up-and-out barrier call is the price of an otherwise same European call. The price of the European call is therefore USD 5.21 + USD 1.40 = USD 6.61. The put-call parity relation gives call – put = forward (with same strikes and maturities). Thus 6.61 – put = 1.50. Thus put = 6.61 – 1.50 = 5.11.

(See Topic 11)

Question from the 2011 FRM practice exam.

5. C The shout call option and lookback call option are clearly wrong, since they grant more rights to the buyer than the European call option. American calls also offer more to the buyer than the European calls.

(See Topic 11)

Question from the 2011 FRM practice exam.

6. C Both

Both are factors affecting prepayment.

(See Topic 13)
Question from the 2011 FRM practice exam.

7. C Both
   I. Correct. MBS' cash flows are like annuities, which are more sensitive to yield curve twist because of reinvestment risk. Normal bond has a lump sum payment at maturity, which implies less reinvestment risk.
   II. Correct. When yield is higher than MBS' coupon rate, the embedded call option is out of the money. It is much the same as a normal bond.
   (See Topic 13)

Question from the 2011 FRM practice exam.

8. B II only
   I. False. Holding MBS is equivalent to holding a similar duration bond and selling a call option. As yield volatility increases, the value of embedded call option increases. Thus the value of MBS decreases.
   II. True. A rise in interest rates reduces the prepayments and hence increases the duration of a MBS.
   (See Topic 13)

Question from the 2010 FRM practice exam.

9. D USD 17 million
   A. Incorrect. This is the minimum.
   B. Incorrect. This is the maximum.
   C. Incorrect. This is the median.
   D. Correct. Conditional VaR is the “mean” of the losses beyond the VaR level.
   (See Topic 1)

Question from the 2010 FRM practice exam.

10. B 0.0665
    The DV01 of a zero-coupon is \( DV01 = \frac{30}{[100 \ (1 + y / 2)^{2T}]} = \frac{30}{[190 \ (1 + 5\% / 2)^{6}]} \)
    = 0.0665.
    (See Topic 7)

Question from the 2010 FRM practice exam.

11. C For each option, use the implied volatility of the most similar option traded on the market.
    The prices obtained with C are the right ones because they correspond to prices at which you could sell or buy the options.
    (See Topic 10)
Question from the 2010 FRM practice exam.

12. C The book has a long position in up-and-out call options.
   
   As the underlying assets’ price increases the up-and-out call options become more vulnerable since they will cease to exist when the barrier is reached. Hence their price decreases. This is negative delta.

   (See Topic 11)

Question from the 2010 FRM practice exam.

13. A I and III

   A call spread replicates a cash-or-nothing option. Such long call spread is constituted by a long call C1 with a strike K - epsilon, and a short call C2 with a strike K + epsilon where epsilon is small. The strategy is market bullish, the delta is always positive so I is true. Furthermore, the vega and gamma can be positive or negative depending on the spot level. When the underlying price is bigger than the strike price, the vega is negative and the gamma as well corresponding to C2’s Greeks. So, II is wrong and III is true.

   (See Topic 11)

Question from the 2009 FRM practice exam.

14. C By smoothing the tail of the distribution, EVT effectively ignores extreme events and losses which can generally be labeled outliers.

   A. Correct.
   B. Correct.
   C. Incorrect. Outliers are a big concern in risk analysis and cannot be ignored.
   D. Correct.

   (See Topic 4)

Question from the 2009 FRM practice exam.

15. A USD 1.6 million

   Because the floating-rate leg has just been reset, its duration is 1. Net duration is $4.5 - 1 = 3.5$ years, or modified duration of $3.5 / 1.05 = 3.33$. The 95% VaR of monthly changes in yields is $1.65 \times 1\% / \sqrt{12} = 0.48\%$. Multiplying, this gives USD $100 \times 0.48\% \times 3.33 = USD 1.588$.

   A. Correct.
   B. Incorrect. This uses a net duration of 4.5 years and ignores the duration of the floating-rate leg.
   C. Incorrect. This is the annual VaR, but should be translated to a monthly horizon.
   D. Incorrect. This is the annual VaR computed by ignoring the duration of the floating-rate leg.

   (See Topic 7)
Question from the 2009 FRM practice exam.

16. D USD 105,000,000

USD 105,000,000 x .076 / 100 = USD 79,800, which is pretty close to the desired DV01 of USD 80,000. To solve for the hedge, solve for F in the equation USD 80,000 = F x .076 / 100, giving F = 105,263,158.

A. Incorrect. Selling this amount would offset a DV01 of only USD 80,000 x .076 / 100 = USD 61.
B. Incorrect. USD 10,500,000 x .076 / 100 = USD 7,980.
C. Incorrect. USD 80,000,000 x .076 / 100 = USD 60,800.
D. Correct.
(See Topic 7)

Question from the 2009 FRM practice exam.

17. B 2.3 years, 7.5%

The calculation of portfolio duration and portfolio yield is based on the proportional weightage of respective company to its duration and yield. The portfolio duration and portfolio yield after mapping the yield from rating matrix is answer B.

This answer reflects the proportion of amount taken as weight to calculate the portfolio duration and portfolio yield.

<table>
<thead>
<tr>
<th>GARP Bond Fund</th>
<th>Amount</th>
<th>Proportion %</th>
<th>Duration</th>
<th>Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co A</td>
<td>600</td>
<td>30%</td>
<td>1.5</td>
<td>6.75%</td>
</tr>
<tr>
<td>Co B</td>
<td>300</td>
<td>15%</td>
<td>4</td>
<td>8.00%</td>
</tr>
<tr>
<td>Co C</td>
<td>200</td>
<td>10%</td>
<td>2.5</td>
<td>7.35%</td>
</tr>
<tr>
<td>AA</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co D</td>
<td>400</td>
<td>20%</td>
<td>4</td>
<td>8.80%</td>
</tr>
<tr>
<td>Co E</td>
<td>350</td>
<td>18%</td>
<td>0.5</td>
<td>6.75%</td>
</tr>
<tr>
<td>A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Co F</td>
<td>150</td>
<td>8%</td>
<td>1.5</td>
<td>8.45%</td>
</tr>
<tr>
<td>Total</td>
<td>2,000</td>
<td>100%</td>
<td>2.30</td>
<td>7.54</td>
</tr>
</tbody>
</table>

A. Incorrect. If the candidate does a simple addition of the duration and the mapped yield, he would get this answer.
B. Correct.
C. Incorrect. Though the portfolio duration is correct, it is arrived by taking simple average of duration. However, if the candidate would take simple average of mapped yield instead of proportion his answer would be 7.7% and not 7.5%.
D. Incorrect. If the candidate averages based on rating classes (3 – AAA/AA/A) instead of the companies, he would get this answer.
(See Topic 7)
Question from the 2009 FRM practice exam.

18. B A deep in the money up and out call option has a negative vega.

Deep in the money up and out call option because as increase in the volatility of such options leads to the increasing chances of option either being knocked out (if the price increases beyond the barrier) or losing its moneyness (if the price falls) and hence the increasing volatility tends to have negative impact on the price of the option.

A. Incorrect. As an option holder can be vega negative as shown above.
B. Correct.
C. Incorrect. Have positive vega as an increase in the volatility would increase the chances of getting towards moneyness and hence positive vega from a holder’s perspective.
D. Incorrect. Have positive vega as an increase in the volatility would increase the chances of getting towards moneyness and hence positive vega from a holder’s perspective.

(See Topic 11)

Question from the 2008 FRM practice exam.


A. Incorrect. Premium Bond will have a higher Base Price and hence higher DV01 than that of Zero Coupon Bond.
B. Correct. DV01 is certain multiple of Dirty Price (which includes Coupons) and not Clean Price. Thus, it is proportional to Base Price, which is Dirty Price. Ordinarily, Premium Bond will have the highest (dirty) price followed by Par Bond and with the least price of Zero Coupon Bond. Hence, DV01 of Premium Bond is the highest while that of Zero Coupon Bond is the lowest.
C. Incorrect. Base Price of Par Bond is higher than that of Zero Coupon Bond and hence, its DV01 cannot be less than that of Zero Coupon Bond.
D. Incorrect. DV01 per USD 100 Face Value is an Absolute Amount of USD based on actual Base Price Change. Ordinarily, Base Price of a Zero Coupon Bond will be lower than that of Par & Premium Bond. Hence, DV01 of Zero Coupon Bond is less than that of Premium Bond of same maturity.

(See Topic 7)

Question from the 2008 FRM practice exam.

20. A 9

You must first recall that the Macauley duration of a zero-coupon bond is equal to its maturity. Then, the modified duration of a zero-coupon bond is: Macauley duration / 1 + i = 10 / 1.10 = 9.09.

A. Correct. The above formula was used correctly. Dmod = Macauley duration / 1 + i.
B. Incorrect. It corresponds to the Macauley duration, not the Modified duration.
C. Incorrect. The denominator used in the formula was i instead of 1+i.
D. Incorrect. All the necessary information is in there.

(See Topic 7)
Question from the 2008 FRM practice exam.

21. A  The market price will be higher than the theoretical price by USD 5.
    A. Correct. Given the same strike price and time to expiration, option market prices that
deviate from those dictated by the Black-Scholes model are going to deviate in the same
amount whether they are for calls or puts.
    B. Incorrect. The deviation will be for the same amount, not a lower amount. Given the
same strike price and time to expiration, option market prices that deviate from those
dicted by the Black-Scholes model are going to deviate in the same amount whether they
are for calls or puts.
    C. Incorrect. The market price will be higher, not lower than the theoretical price. Given
the same strike price and time to expiration, option market prices that deviate from those
dicted by the Black-Scholes model are going to deviate in the same amount whether they
are for calls or puts.
    D. Incorrect. The market price will be higher, not lower than the theoretical price. Given
the same strike price and time to expiration, option market prices that deviate from those
dicted by the Black-Scholes model are going to deviate in the same amount whether they
are for calls or puts.

(See Topic 10)

Question from the 2008 FRM practice exam.

22. B  I and IV
    I. Correct. The payoff on look-back options depends on the maximum or minimum
underlying price achieved during the life of the option. The option holder is guaranteed
the most favorable rate during the life of the option. As a result, the premium is
substantially higher than plain vanilla options.
    II. Incorrect. A barrier option is extinguished or created only when the barrier is touched.
For example, an up-and-in call option would only be created if at some point during
the option’s life the price of the underlying exceeded the barrier, if it failed to do so it could
not be exercised regardless of whether it finished in-the-money or not. Similarly, a down-
and-out put option would automatically be extinguished if, during the option’s, life the
underlying asset’s price fell below the barrier. Barrier options are always less expensive
than plain vanilla options as there is always a probability that the options will be knocked
out or not be knocked in.
    III. Incorrect. The pay-off for Asian options is based on the average price of the underlying
asset over the life of the option and not a set strike price. Asian options are cheaper than
plain vanilla options as average prices are less volatile than day-to-day prices.
    IV. Correct. A chooser option has the feature that after a specified period of time, the holder
can choose to decide whether the option is a put or call. As a result of this increased
flexibility, chooser options are more expensive than plain vanilla options.

(See Topic 11)

Question from the 2008 FRM practice exam.

23. C  Binary options.
    A. Incorrect. The payoff profile of a chooser option is continuous.
    B. Incorrect. The payoff profile of a barrier option is continuous.
    C. Correct. The binary option is the only one that produces discontinuous payoff profiles
because it pays one price at the expiration if the asset value is above the strike price and
nothing if the asset price is below the strike price.
    D. Incorrect. The payoff profile of a lookback option is continuous.

(See Topic 11)
Question from the 2008 FRM practice exam.

24. A Asian options are more difficult to hedge because they have more extreme gamma towards expiration.
   A. Incorrect. Asian options are easier to hedge because the payoff becomes progressively more certain as we approach maturity.
   B. Correct. Barrier options are more difficult to hedge because delta is liable to be discontinuous at the barrier.
   C. Correct. The approach of static options replication is to find a portfolio of regular options whose value matches the value of the exotic option on some boundary.
   D. Correct. The portfolio constructed using static option replication must be unwound when any part of the boundary is reached.
   (See Topic 11)

Question from the 2008 FRM practice exam.

25. A Interest-only strips (IO).
   A. Correct. If interest rates fall, IO strips will decrease in value, the other 3 securities will increase in value.
   B. Incorrect. If interest rates fall, IO strips will decrease in value, the other 3 securities will increase in value.
   C. Incorrect. If interest rates fall, IO strips will decrease in value, the other 3 securities will increase in value.
   D. Incorrect. If interest rates fall, IO strips will decrease in value, the other 3 securities will increase in value.

   If interest rates fall, mortgage prepayments will accelerate. PO investors will receive their payments earlier than anticipated. Therefore PO strips will increase in value.

   If interest rates fall, IO strips will decrease in value. This is due to increased mortgage prepayments which will cause the outstanding principal to shrink, that means a decrease in the interest payments.
   (See Topic 13)

Question from the 2008 FRM practice exam.

26. B Y is undervalued.
   A. Incorrect. The OAS of X bond is equal to the comparable option free bond while the option cost is also higher than the Y bond.
   B. Correct. The OAS of the bonds should be compared with the nominal spread on comparable option free bonds. Bonds with higher OAS and low option cost are undervalued and should be bought.
   C. Incorrect. X is not undervalued with a comparable OAS to the nominal spread of the option free bonds.
   D. Incorrect. Y is undervalued and should be bought.
   (See Topic 15)
Book 1 Formulas

Market Risk Measurement and Management

profit/loss data: \( P/L_t = P_t + D_t - P_{t-1} \)

arithmetic return: \( \tau_t = \frac{P_t + D_t - P_{t-1}}{P_{t-1}} = \frac{P_t + D_t}{P_{t-1}} - 1 \)

geometric return: \( R_t = \ln \left( \frac{P_t + D_t}{P_{t-1}} \right) \)

delta-normal VaR: \( \text{VaR}(\alpha\%) = (-\mu_t + \sigma_t \times z_\alpha) \times P_{t-1} \)

lognormal VaR: \( \text{VaR}(\alpha\%) = P_{t-1} \times \left(1 - e^{\mu_R - \sigma_R \times z_\alpha} \right) \)

standard error of a quantile: \( \text{se}(q) = \frac{\sqrt{p(1-p)/n}}{f(q)} \)

age-weighted historic simulation: \( w(i) = \frac{\lambda^{i-1}(1-\lambda)}{1-\lambda^n} \)

generalized extreme value (GEV) distribution:

\[
F(X | \xi, \mu, \sigma) = \exp \left[ - \left( 1 + \xi \times \frac{x - \mu}{\sigma} \right)^{-1/\xi} \right] \text{ if } \xi \neq 0
\]

\[
F(X | \xi, \mu, \sigma) = \exp \left[ - \exp \left( \frac{x - \mu}{\sigma} \right) \right] \text{ if } \xi = 0
\]

generalized Pareto distribution:

\[
1 - \left[ 1 + \frac{\xi x}{\beta} \right]^{-1/\xi} \text{ if } \xi \neq 0
\]

\[
1 - \exp \left[ - \frac{x}{\beta} \right] \text{ if } \xi = 0
\]
VaR using POT parameters: \( \text{VaR} = u + \frac{\beta}{\xi} \left[ \frac{n}{N_u} (1 - \text{confidence level}) \right]^{-\xi} - 1 \)

expected shortfall using POT parameters: \( \text{ES} = \frac{\text{VaR}}{1 - \xi} + \frac{\beta - \xi u}{1 - \xi} \)

unconditional coverage test statistic: \( \text{LR}_{uc} = -2 \ln \{ (1 - P)^T N p^N \} + 2 \ln \{ (1 - (N/T))^T N (N/T)^N \} \)

yield-based DV01 = \( \left( \frac{1}{10,000} \right) \times \left( \frac{1}{1 + \text{periodic yield}} \right) \times \left( \text{sum of time-weighted present values of the bond's cash flows} \right) \)

modified duration = \( \left( \frac{1}{P} \right) \times \left( \frac{1}{1 + \text{periodic yield}} \right) \times \left( \text{sum of the time-weighted present values of the bond’s cash flows} \right) \)

Macaulay duration = \( (1 + \text{periodic yield}) \times \text{modified duration} \)

put-call parity: \( c - p = S - \text{PV}(X) \)

mortgage payment factor: \( \text{payment}_{\text{monthly}} = \text{original loan balance} \times \frac{r \times (1 + r)^T}{(1 + r)^T - 1} \)

where:
\( r \) = monthly interest rate
\( T \) = loan term (in months)

single monthly mortality rate: \( \text{SMM} = 1 - (1 - \text{CPR})^{1/12} \)

effective duration: \( -\frac{1}{P} \times \frac{P_{+\Delta y} - P_{-\Delta y}}{2 \times \Delta y} \)

convexity approximation: \( \frac{1}{P} \times \frac{P_{+\Delta y} - P_{-\Delta y} - 2 \times P}{\Delta y^2} \)

excess servicing:
loan (note) rate – servicing fees – guarantee fees – pass-through pool coupon rate
bond equivalent yield: \[ BEY = 2[(1 + i_M)^6 - 1] \]
where:
\[ i_M = \text{monthly mortgage yield} \]

theoretical value of mortgage security:
\[ \text{theoretical value} = \frac{\text{PV[path(1)]} + \text{PV[path(2)]} + \ldots + \text{PV[path(N)]}}{N} \]
where:
\[ N = \text{number of interest rate paths} \]

option cost = zero-volatility spread – OAS

mean standard error: \[ MSE = \sqrt{\frac{\text{variance of the trial values}}{\text{number of trials}}} \]
**Using the Cumulative Z-Table**

**Probability Example**

Assume that the annual earnings per share (EPS) for a large sample of firms is normally distributed with a mean of $5.00 and a standard deviation of $1.50. What is the approximate probability of an observed EPS value falling between $3.00 and $7.25?

If \( \text{EPS} = x = 7.25 \), then \( z = (x - \mu) / \sigma = (7.25 - 5.00) / 1.50 = 1.50 \)

If \( \text{EPS} = x = 3.00 \), then \( z = (x - \mu) / \sigma = (3.00 - 5.00) / 1.50 = -1.33 \)

For \( z \)-value of 1.50: Use the row headed 1.5 and the column headed 0 to find the value 0.9332. This represents the area under the curve to the left of the critical value 1.50.

For \( z \)-value of -1.33: Use the row headed 1.3 and the column headed 3 to find the value 0.9082. This represents the area under the curve to the left of the critical value -1.33. The area to the left of -1.33 is \( 1 - 0.9082 = 0.0918 \).

The area between these critical values is 0.9332 - 0.0918 = 0.8414, or 84.14%.

**Hypothesis Testing – One-Tailed Test Example**

A sample of a stock’s returns on 36 non-consecutive days results in a mean return of 2.0%. Assume the population standard deviation is 20.0%. Can we say with 95% confidence that the mean return is greater than 0%?

\[
H_0: \mu \leq 0.0\% \quad H_A: \mu > 0.0\%
\]

The test statistic \( z \)-statistic = \( \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \) = \( \frac{2.0 - 0.0}{20.0 / 6} \) = 0.60.

The significance level = 1.0 - 0.95 = 0.05, or 5%.

Since this is a one-tailed test with an alpha of 0.05, we need to find the value 0.95 in the cumulative \( z \)-table. The closest value is 0.9505, with a corresponding critical \( z \)-value of 1.65. Since the test statistic is less than the critical value, we fail to reject \( H_0 \).

**Hypothesis Testing – Two-Tailed Test Example**

Using the same assumptions as before, suppose that the analyst now wants to determine if he can say with 99% confidence that the stock’s return is not equal to 0.0%.

\[
H_0: \mu = 0.0\% \quad H_A: \mu \neq 0.0\%
\]

The test statistic \( z \)-value = \( \frac{2.0 - 0.0}{20.0 / 6} \) = 0.60.

The significance level = 1.0 - 0.99 = 0.01, or 1%.

Since this is a two-tailed test with an alpha of 0.01, there is a 0.005 rejection region in both tails. Thus, we need to find the value 0.995 (1.0 - 0.005) in the table. The closest value is 0.9951, which corresponds to a critical \( z \)-value of 2.58. Since the test statistic is less than the critical value, we fail to reject \( H_0 \) and conclude that the stock’s return equals 0.0%.

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### Cumulative Z-Table

\( P(Z \leq z) = N(z) \) for \( z \geq 0 \)

\( P(Z \leq -z) = 1 - N(z) \)

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<th>( z )</th>
<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
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<th>0.07</th>
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| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8880 | 0.8889 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |

| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9846 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9947 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9989 | 0.9990 |
### Alternative Z-Table

\( P(Z \leq z) = N(z) \) for \( z \geq 0 \)

\( P(Z \leq -z) = 1 - N(z) \)

<table>
<thead>
<tr>
<th>( z )</th>
<th>0.00</th>
<th>0.01</th>
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### Student's T-Distribution

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